VERSION 1 CAPS

GRADE 10 MATHEMATICS

WRITTEN BY VOLUNTEERS

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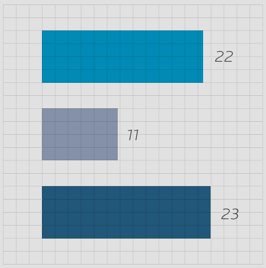

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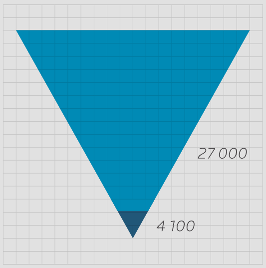

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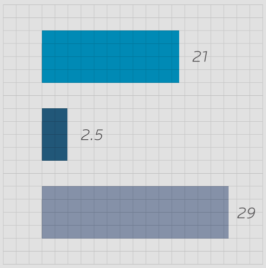

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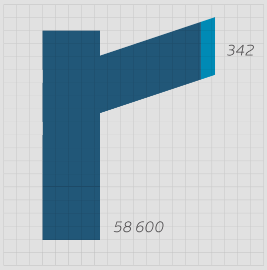

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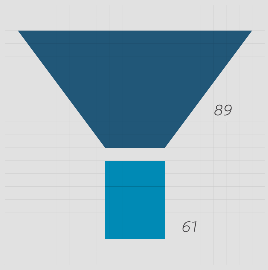

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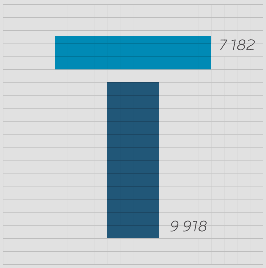

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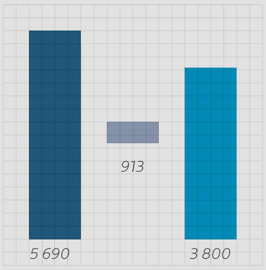

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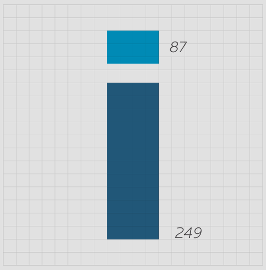

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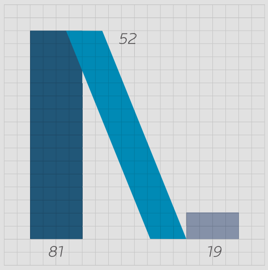

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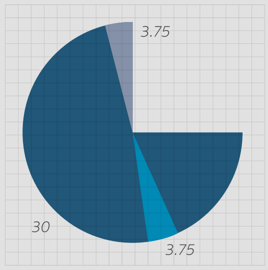

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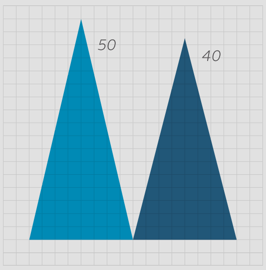

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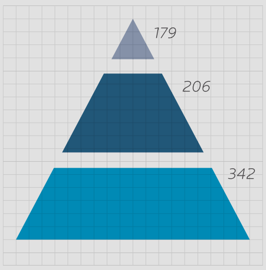

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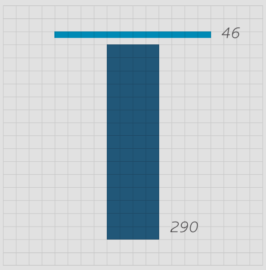

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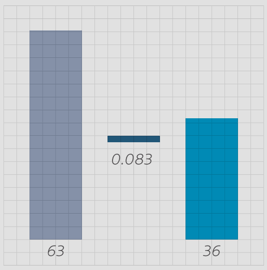

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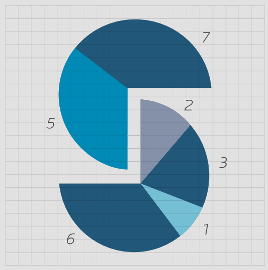

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Everything Maths

Grade 10 Mathematics

Version 1 – CAPS

by Siyavula and volunteers

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Everything Maths

Mathematics is commonly thought of as being about numbers but mathematics is actually a language! Mathematics is the language that nature speaks to us in. As we learn to understand and speak this language, we can discover many of nature's secrets. Just as understanding someone's language is necessary to learn more about them, mathematics is required to learn about all aspects of the world – whether it is physical sciences, life sciences or even finance and economics.

The great writers and poets of the world have the ability to draw on words and put them together in ways that can tell beautiful or inspiring stories. In a similar way, one can draw on mathematics to explain and create new things. Many of the modern technologies that have enriched our lives are greatly dependent on mathematics. DVDs, Google searches, bank cards with PIN numbers are just some examples. And just as words were not created specifically to tell a story but their existence enabled stories to be told, so the mathematics used to create these technologies was not developed for its own sake, but was available to be drawn on when the time for its application was right.

There is in fact not an area of life that is not affected by mathematics. Many of the most sought after careers depend on the use of mathematics. Civil engineers use mathematics to determine how to best design new structures; economists use mathematics to describe and predict how the economy will react to certain changes; investors use mathematics to price certain types of shares or calculate how risky particular investments are; software developers use mathematics for many of the algorithms (such as Google searches and data security) that make programmes useful.

But, even in our daily lives mathematics is everywhere – in our use of distance, time and money. Mathematics is even present in art, design and music as it informs proportions and musical tones. The greater our ability to understand mathematics, the greater our ability to appreciate beauty and everything in nature. Far from being just a cold and abstract discipline, mathematics embodies logic, symmetry, harmony and technological progress. More than any other language, mathematics is everywhere and universal in its application.

See introductory video by Dr. Mark Horner: ▶ VMiwd at www.everythingmaths.co.za

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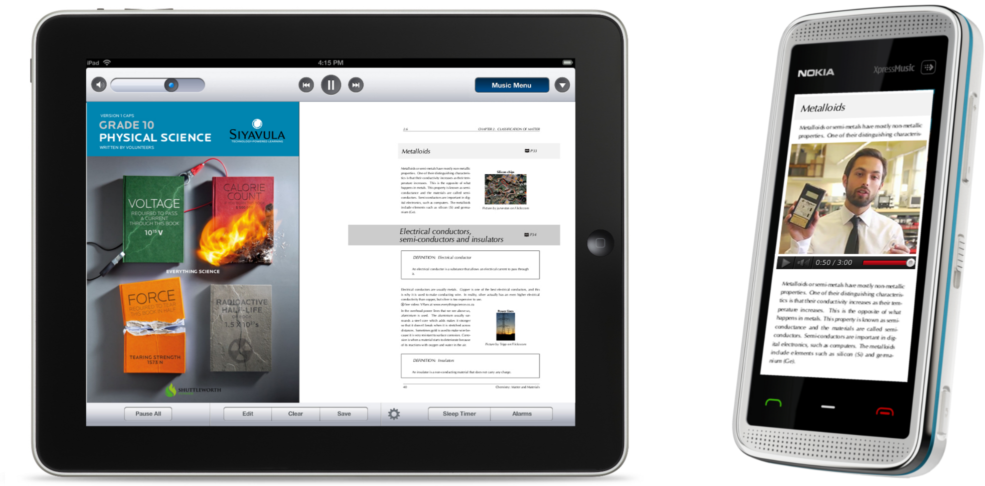

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www (A123) Go directly to a section

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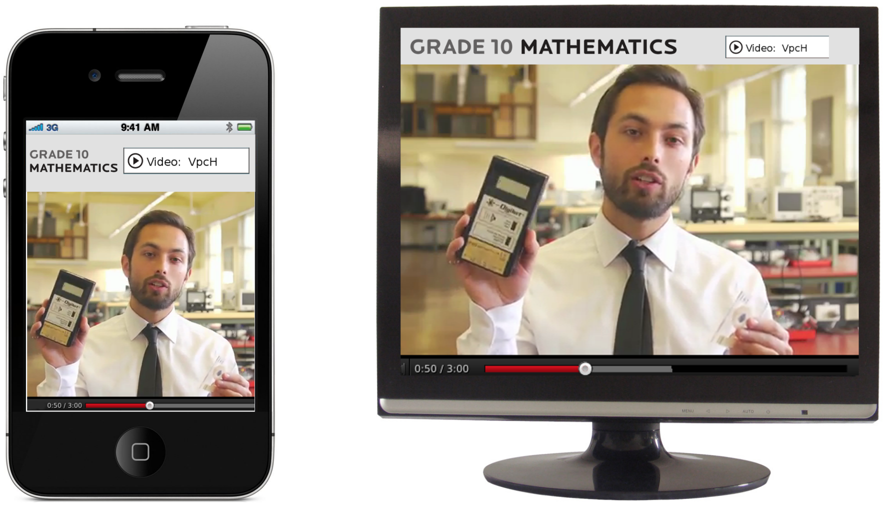

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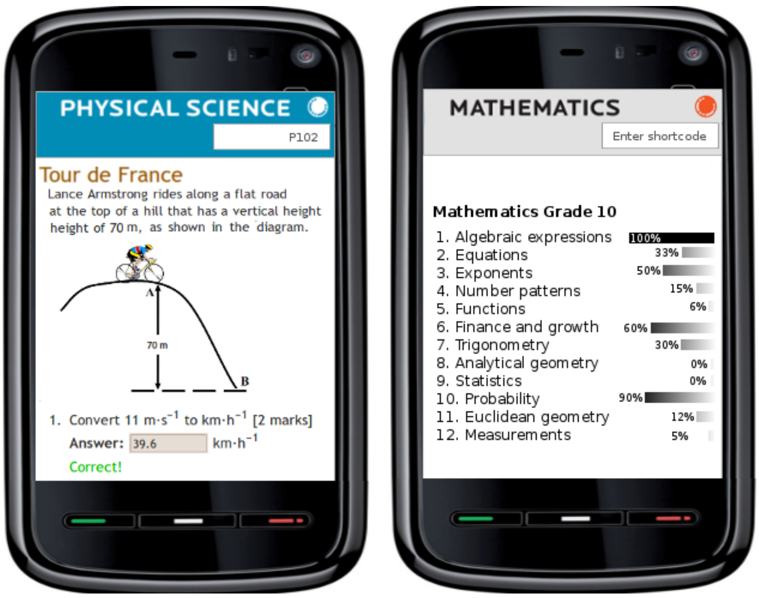

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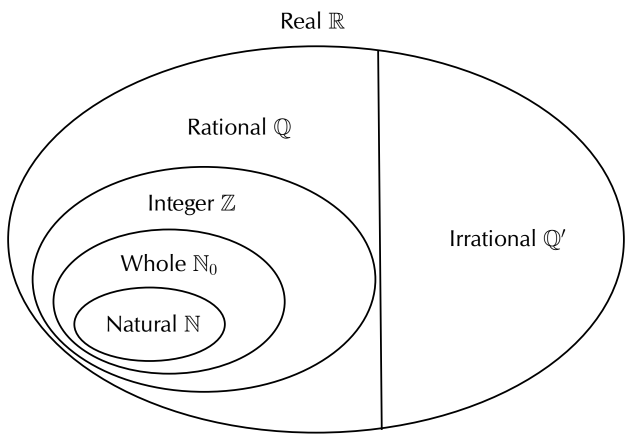
Contents

Algebraic expressions 1

1.1 The real number system EMAA



We use the following definitions:

- $\mathbb{N}$: natural numbers are $\{1;\ 2;\ 3;\ \ldots\}$
- $\mathbb{N}_0$: whole numbers are $\{0;\ 1;\ 2;\ 3;\ \ldots\}$
- $\mathbb{Z}$: integers are $\{\ldots;\ -3;\ -2;\ -1;\ 0;\ 1;\ 2;\ 3;\ \ldots\}$

Video: VMabo at www.everythingmaths.co.za

1.2 Rational and irrational numbers EMAB

DEFINITION: Rational number

A rational number ($\mathbb{Q}$) is any number which can be written as:

$$\frac{a}{b}$$

where a and b are integers and $b \neq 0$.

The following numbers are all rational numbers:

$$\frac{10}{1};\ \frac{21}{7};\ \frac{-1}{-3};\ \frac{10}{20};\ \frac{-3}{6}$$

We see that all numerators and all denominators are integers. This means that all integers are rational numbers, because they can be written with a denominator of 1.

DEFINITION: Irrational numbers

Irrational numbers ($\mathbb{Q}'$) are numbers that cannot be written as a fraction with the numerator and denominator as integers.

Examples of irrational numbers:

$$\sqrt{2};\ \sqrt{3};\ \sqrt[3]{4};\ \pi;\ \frac{1+\sqrt{5}}{2}$$

These are not rational numbers, because either the numerator or the denominator is not an integer.

Decimal numbers

EMAC

All integers and fractions with integer numerators and denominators are rational numbers.

You can write any rational number as a decimal number but not all decimal numbers are rational numbers. These types of decimal numbers are rational numbers:

- Decimal numbers that end (or terminate). For example, the fraction $\frac{4}{10}$ can be written as $0{,}4$.
- Decimal numbers that have a repeating single digit. For example, the fraction $\frac{1}{3}$ can be written as $0{,}\dot{3}$ or as $0{,}\bar{3}$. The dot and bar notations are equivalent and both represent recurring 3's, i.e. $0{,}\dot{3} = 0{,}\bar{3} = 0{,}333\ldots$.
- Decimal numbers that have a recurring pattern of multiple digits. For example, the fraction $\frac{2}{11}$ can also be written as $0{,}\overline{18}$. The bar represents a recurring pattern of 1 and 8's i.e. $0{,}\overline{18} = 0{,}181818\ldots$.

Notation: You can use a dot or a bar over the repeated numbers to indicate that the decimal is a recurring decimal. If the bar covers more than one number, then all numbers beneath the bar are recurring.

If you are asked to identify whether a number is rational or irrational, first write the number in decimal form. If the number terminates then it is rational. If it goes on forever, then look for a repeated pattern of digits. If there is no repeated pattern, then the number is irrational.

When you write irrational numbers in decimal form, you may continue writing them for many, many decimal places. However, this is not convenient and it is often necessary to round off.

Example 1: *Rational and irrational numbers*

QUESTION

Which of the following are not rational numbers?

1. $\pi = 3{,}14159265358979323846264338327950288419716939937510\ldots$
2. $1{,}4$
3. $1{,}618033989\ldots$
4. 100
5. $1{,}7373737373\ldots$
6. $0{,}\overline{02}$

SOLUTION

1. Irrational, decimal does not terminate and has no repeated pattern.
2. Rational, decimal terminates.
3. Irrational, decimal does not terminate and has no repeated pattern.
4. Rational, all integers are rational.
5. Rational, decimal has repeated pattern.
6. Rational, decimal has repeated pattern.

Converting terminating decimals into rational numbers EMAD

A decimal number has an integer part and a fractional part. For example, $10{,}589$ has an integer part of 10 and a fractional part of $0{,}589$ because $10 + 0{,}589 = 10{,}589$. The fractional part can be written as a rational number, i.e. with a numerator and denominator that are integers.

Each digit after the decimal point is a fraction with a denominator in increasing powers of 10.
For example,

- $0{,}1$ is $\frac{1}{10}$
- $0{,}01$ is $\frac{1}{100}$
- $0{,}001$ is $\frac{1}{1\ 000}$

This means that

$$\begin{aligned}10{,}589 &= 10 + \frac{5}{10} + \frac{8}{100} + \frac{9}{1\ 000}\\ &= \frac{10\ 000}{1\ 000} + \frac{500}{1\ 000} + \frac{80}{1\ 000} + \frac{9}{1\ 000}\\ &= \frac{10\ 589}{1\ 000}\end{aligned}$$

Converting recurring decimals into rational numbers EMAE

When the decimal is a recurring decimal, a bit more work is needed to write the fractional part of the decimal number as a fraction.

Example 2: *Converting decimal numbers to fractions*

QUESTION

Write $0{,}\dot{3}$ *in the form* $\frac{a}{b}$ *(where* a *and* b *are integers).*

SOLUTION

Step 1 : ***Define an equation***

$$\text{Let } x = 0{,}33333\ldots$$

Step 2 : ***Multiply by*** 10 ***on both sides***

$$10x = 3{,}33333\ldots$$

Step 3 : ***Subtract the first equation from the second equation***

$$9x = 3$$

Step 4 : ***Simplify***

$$x = \frac{3}{9} = \frac{1}{3}$$

Example 3: *Converting decimal numbers to fractions*

QUESTION

Write $5{,}\dot{4}3\dot{2}$ *as a rational fraction.*

SOLUTION

Step 1 : ***Define an equation***

$$x = 5{,}432432432\ldots$$

Step 2 : ***Multiply by*** 1000 ***on both sides***

$$1\ 000x = 5\ 432{,}432432432\ldots$$

Step 3 : ***Subtract the first equation from the second equation***

$$999x = 5\ 427$$

Step 4 : ***Simplify***

$$x = \frac{5\ 427}{999} = \frac{201}{37} = 5\frac{16}{37}$$

In the first example, the decimal was multiplied by 10 and in the second example, the decimal was multiplied by $1\ 000$. This is because there was only one digit recurring (i.e. 3) in the first example, while there were three digits recurring (i.e. 432) in the second example.

In general, if you have one digit recurring, then multiply by 10. If you have two digits recurring, then multiply by 100. If you have three digits recurring, then multiply by $1\ 000$ and so on.

Not all decimal numbers can be written as rational numbers. Why? Irrational decimal numbers like $\sqrt{2} = 1{,}4142135\ldots$ cannot be written with an integer numerator and denominator, because they do not have a pattern of recurring digits and they do not terminate. However, when possible, you should try to use rational numbers or fractions instead of decimals.

Exercise 1 - 1

1. State whether the following numbers are rational or irrational. If the number is rational, state whether it is a natural number, whole number or an integer:
   (a) $-\frac{1}{3}$
   (b) $0{,}651268962154862\ldots$
   (c) $\frac{\sqrt{9}}{3}$
   (d) π^2
2. If a is an integer, b is an integer and c is irrational, which of the following are rational numbers?
   (a) $\frac{5}{6}$
   (b) $\frac{a}{3}$
   (c) $\frac{-2}{b}$
   (d) $\frac{1}{c}$
3. For which of the following values of a is $\frac{a}{14}$ rational or irrational?
   (a) 1
   (b) -10
   (c) $\sqrt{2}$
   (d) $2{,}1$
4. Write the following as fractions:
   (a) $0{,}1$
   (b) $0{,}12$
   (c) $0{,}58$
   (d) $0{,}2589$
5. Write the following using the recurring decimal notation:
   (a) $0{,}11111111\ldots$

(b) $0{,}1212121212\ldots$

(c) $0{,}123123123123\ldots$

(d) $0{,}11414541454145\ldots$

6. Write the following in decimal form, using the recurring decimal notation:

(a) $\frac{2}{3}$

(b) $1\frac{3}{11}$

(c) $4\frac{5}{6}$

(d) $2\frac{1}{9}$

7. Write the following decimals in fractional form:

(a) $0{,}\dot{5}$

(b) $0{,}6\dot{3}$

(c) $5{,}\overline{31}$

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(1.) 023j (2.) 00bb (3.) 00bc (4.) 00bd (5.) 00be

(6.) 00bf (7.) 00bg

1.3 Rounding off EMAF

Rounding off a decimal number to a given number of decimal places is the quickest way to approximate a number. For example, if you wanted to round off $2{,}6525272$ to three decimal places, you would:

- count three places after the decimal and place a $|$ between the third and fourth numbers
- round up the third digit if the fourth digit is greater than or equal to 5
- leave the third digit unchanged if the fourth digit is less than 5
- if the third digit is 9 and needs to be round up, then the 9 becomes a 0 and the second digit rounded up

So, since the first digit after the | is a 5, we must round up the digit in the third decimal place to a 3 and the final answer of $2{,}6525272$ rounded to three decimal places is $2{,}653$.

Example 4: *Rounding off*

QUESTION

Round off the following numbers to the indicated number of decimal places:

1. $\frac{120}{99} = 1{,}\dot{1}\dot{2}$ *to* 3 *decimal places.*
2. $\pi = 3{,}141592653\ldots$ *to* 4 *decimal places.*
3. $\sqrt{3} = 1{,}7320508\ldots$ *to* 4 *decimal places.*
4. $2{,}78974526$ *to* 3 *decimal places.*

SOLUTION

Step 1 : ***Mark off the required number of decimal places***

1. $\frac{120}{99} = 1{,}212|121212\ldots$
2. $\pi = 3{,}1415|92653\ldots$
3. $\sqrt{3} = 1{,}7320|508\ldots$
4. $2{,}789|74526$

Step 2 : ***Check the next digit to see if you must round up or round down***

1. The last digit of $\frac{120}{99} = 1{,}212|1212121\dot{1}\dot{2}$ must be rounded down.
2. The last digit of $\pi = 3{,}1415|92653\ldots$ must be rounded up.
3. The last digit of $\sqrt{3} = 1{,}7320|508\ldots$ must be rounded up.
4. The last digit of $2{,}789|74526$ must be rounded up.
   Since this is a 9, we replace it with a 0 and round up
   the second last digit.

Step 3 : ***Write the final answer***

1. $\frac{120}{99} = 1{,}212$ rounded to 3 decimal places.
2. $\pi = 3{,}1416$ rounded to 4 decimal places.
3. $\sqrt{3} = 1{,}7321$ rounded to 4 decimal places.
4. $2{,}790$.

Exercise 1 - 2

Round off the following to 3 decimal places:

1. $12{,}56637061\ldots$
2. $3{,}31662479\ldots$
3. $0{,}26666666\ldots$
4. $1{,}912931183\ldots$
5. $6{,}32455532\ldots$
6. $0{,}05555555\ldots$

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(1.-6.) 00bh

1.4 Estimating surds EMAG

If the n^{th} root of a number cannot be simplified to a rational number, we call it a surd. For example, $\sqrt{2}$ and $\sqrt[3]{6}$ are surds, but $\sqrt{4}$ is not a surd because it can be simplified to the rational number 2.

In this chapter we will look at surds of the form $\sqrt[n]{a}$ where a is any positive number, for example, $\sqrt{7}$ or $\sqrt[3]{5}$. It is very common for n to be 2, so we usually do not write $\sqrt[2]{a}$. Instead we write the surd as just $\sqrt{a}$.

It is sometimes useful to know the approximate value of a surd without having to use a calculator. For example, we want to be able to estimate where a surd like $\sqrt{3}$ is on the number line. From a calculator we know that $\sqrt{3}$ is equal to $1{,}73205\ldots$. It is easy to see that $\sqrt{3}$ is above 1 and below 2. But to see this for other surds like $\sqrt{18}$ without using a calculator, you must first understand the following:

> If a and b are positive whole numbers, and $a < b$, then $\sqrt[n]{a} < \sqrt[n]{b}$.

A perfect square is the number obtained when an integer is squared. For example, 9 is a perfect square since $3^2 = 9$.
Similarly, a perfect cube is a number which is the cube of an integer. For example, 27 is a perfect cube, because $3^3 = 27$.

Consider the surd $\sqrt[3]{52}$. It lies somewhere between 3 and 4, because $\sqrt[3]{27} = 3$ and $\sqrt[3]{64} = 4$ and 52 is between 27 and 64.

Example 5: *Estimating surds*

QUESTION

Find the two consecutive integers such that $\sqrt{26}$ lies between them. (Remember that consecutive integers are two integers that follow one another on the number line, for example, 5 and 6 or 8 and 9).

SOLUTION

Step 1 : ***Use perfect squares to estimate the lower integer***

$5^2 = 25$. Therefore $5 < \sqrt{26}$.

Step 2 : ***Use perfect squares to estimate the upper integer***

$6^2 = 36$. Therefore $\sqrt{26} < 6$.

Step 3 : ***Write the final answer***

$5 < \sqrt{26} < 6$.

Example 6: *Estimating surds*

QUESTION

Find the two consecutive integers such that $\sqrt[3]{49}$ lies between them.

SOLUTION

Step 1 : ***Use perfect cubes to estimate the lower integer***

$3^3 = 27$, therefore $3 < \sqrt[3]{49}$.

Step 2 : ***Use perfect cubes to estimate the upper integer***

$4^3 = 64$, therefore $\sqrt[3]{49} < 4$.

Step 3 : ***Write the answer***

$3 < \sqrt[3]{49} < 4$

Step 4 : ***Check the answer by cubing all terms in the inequality and then simplify***

$27 < 49 < 64$. This is true, so $\sqrt[3]{49}$ lies between 3 and 4.

Exercise 1 - 3

Determine between which two consecutive integers the following numbers lie, without using a calculator:

1. $\sqrt{18}$
2. $\sqrt{29}$
3. $\sqrt[3]{5}$
4. $\sqrt[3]{79}$

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(1.-4.) 00bi

1.5 Products

EMAH

Mathematical expressions are just like sentences and their parts have special names. You should be familiar with the following words used to describe the parts of mathematical expressions.

$$3x^2 + 7xy - 5^3 = 0$$

Name	Examples
term	$3x^2;\ 7xy;\ -5^3$
expression	$3x^2 + 7xy - 5^3$
coefficient	$3;\ 7$
exponent	$2;\ 1;\ 3$
base	$x;\ y;\ 5$
constant	$3;\ 7;\ 5$
variable	$x;\ y$
equation	$3x^2 + 7xy - 5^3 = 0$

Multiplying a monomial and a binomial

EMAI

A monomial is an expression with one term, for example, $3x$ or y^2. A binomial is an expression with two terms, for example, $ax + b$ or $cx + d$.

Example 7: *Simplifying brackets*

QUESTION

Simplify: $2a(a-1)-3(a^2-1)$.

SOLUTION

$$\begin{aligned}2a(a-1)-3(a^2-1) &= 2a(a)+2a(-1)+(-3)(a^2)+(-3)(-1)\\ &= 2a^2-2a-3a^2+3\\ &= -a^2-2a+3\end{aligned}$$

Multiplying two binomials EMAJ

Here we multiply (or expand) two linear binomials:

$$(ax+b)(cx+d)$$

$$\begin{aligned}(ax+b)(cx+d) &= (ax)(cx)+(ax)d+b(cx)+bd\\ &= acx^2+adx+bcx+bd\\ &= acx^2+x(ad+bc)+bd\end{aligned}$$

Example 8: *Multiplying two binomials*

QUESTION

Find the product: $(3x - 2)(5x + 8)$.

SOLUTION

$$\begin{aligned}(3x - 2)(5x + 8) &= (3x)(5x) + (3x)(8) + (-2)(5x) + (-2)(8) \\ &= 15x^2 + 24x - 10x - 16 \\ &= 15x^2 + 14x - 16\end{aligned}$$

The product of two identical binomials is known as the square of the binomial and is written as:

$$(ax + b)^2 = a^2x^2 + 2abx + b^2$$

If the two terms are of the form $ax + b$, and $ax - b$ then their product is:

$$(ax + b)(ax - b) = a^2x^2 - b^2$$

This product yields the difference of two squares.

Multiplying a binomial and a trinomial EMAK

A trinomial is an expression with three terms, for example, $ax^2 + bx + c$. Now we learn how to multiply a binomial and a trinomial.

To find the product of a binomial and a trinomial, multiply out the brackets:

$$(A + B)(C + D + E) = A(C + D + E) + B(C + D + E)$$

Video: MG10062 at www.everythingmaths.co.za

Example 9: *Multiplying a binomial and a trinomial*

QUESTION

Find the product: $(x-1)(x^2-2x+1)$.

SOLUTION

Step 1 : ***Expand the bracket***

$$\begin{aligned}(x-1)(x^2-2x+1) &= x(x^2-2x+1)-1(x^2-2x+1)\\ &= x^3-2x^2+x-x^2+2x-1\end{aligned}$$

Step 2 : ***Simplify***

$$= x^3-3x^2+3x-1$$

Exercise 1 - 4

Expand the following products:

1. $2y(y+4)$

2. $(y+5)(y+2)$

3. $(2-t)(1-2t)$

4. $(x-4)(x+4)$

5. $(2p+9)(3p+1)$

6. $(3k-2)(k+6)$

7. $(s+6)^2$

8. $-(7-x)(7+x)$

9. $(3x-1)(3x+1)$

10. $(7k+2)(3-2k)$

11. $(1-4x)^2$

12. $(-3-y)(5-y)$

13. $(8-x)(8+x)$

14. $(9+x)^2$

15. $(-2y^2-4y+11)(5y-12)$

16. $(7y^2-6y-8)(-2y+2)$

17. $(10y+3)(-2y^2-11y+2)$

18. $(-12y-3)(12y^2-11y+3)$

19. $(-10)(2y^2+8y+3)$

20. $(2y^6+3y^5)(-5y-12)$

21. $(-7y+11)(-12y+3)$

22. $(7y+3)(7y^2+3y+10)$

23. $9(8y^2-2y+3)$

24. $(-6y^4+11y^2+3y)(y+4)(y-4)$

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(1.-6.) 00bj (7.-12.) 00bk (13.-18.) 00bm (19.-24.) 00bn

1.6 Factorisation EMAL

Factorisation is the opposite process of expanding brackets. For example, expanding brackets would require $2(x+1)$ to be written as $2x+2$. Factorisation would be to start with $2x+2$ and to end up with $2(x+1)$.

$$\begin{tikzcd} 2(x+1) \arrow[r, bend right, "\text{expanding}"'] & 2x+2 \arrow[l, bend right, "\text{factorising}"'] \end{tikzcd}$$

The two expressions $2(x+1)$ and $2x+2$ are equivalent; they have the same value for all values of x.

In previous grades, we factorised by taking out a common factor and using difference of squares.

Video: VMabz at www.everythingmaths.co.za

Common factors EMAM

Factorising based on common factors relies on there being factors common to all the terms.

For example, $2x - 6x^2$ can be factorised as follows:

$$2x - 6x^2 = 2x(1 - 3x)$$

Example 10: *Factorising using a switch around in brackets*

QUESTION

Factorise: $5(a-2) - b(2-a)$.

SOLUTION

Use a "switch around" strategy to find the common factor.
Notice that $2 - a = -(a - 2)$

$$\begin{aligned} 5(a-2) - b(2-a) &= 5(a-2) - [-b(a-2)] \\ &= 5(a-2) + b(a-2) \\ &= (a-2)(5+b) \end{aligned}$$

Difference of two squares

EMAN

We have seen that

$$(ax+b)(ax-b) \text{ can be expanded to } a^2x^2 - b^2$$

Therefore

$$a^2x^2 - b^2 \text{ can be factorised as } (ax+b)(ax-b)$$

For example, $x^2 - 16$ can be written as $x^2 - 4^2$ which is a difference of two squares. Therefore, the factors of $x^2 - 16$ are $(x - 4)$ and $(x + 4)$.

To spot a difference of two squares, look for expressions:

- consisting of two terms;
- with terms that have different signs (one positive, one negative);
- with each term a perfect square.

For example: $a^2 - 1$; $4x^2 - y^2$; $-49 + p^4$.

Example 11: *The difference of two squares*

QUESTION

Factorise: $3a(a^2 - 4) - 7(a^2 - 4)$.

SOLUTION

Step 1 : ***Take out the common factor*** $(a^2 - 4)$

$$3a(a^2 - 4) - 7(a^2 - 4) = (a^2 - 4)(3a - 7)$$

Step 2 : ***Factorise the difference of two squares*** $(a^2 - 4)$

$$(a^2 - 4)(3a - 7) = (a - 2)(a + 2)(3a - 7)$$

Exercise 1 - 5

Factorise:

1. $2l + 2w$
2. $12x + 32y$
3. $6x^2 + 2x + 10x^3$
4. $2xy^2 + xy^2z + 3xy$
5. $-2ab^2 - 4a^2b$
6. $7a + 4$
7. $20a - 10$
8. $18ab - 3bc$
9. $12kj + 18kq$
10. $16k^2 - 4$
11. $3a^2 + 6a - 18$
12. $-12a + 24a^3$
13. $-2ab - 8a$
14. $24kj - 16k^2j$
15. $-a^2b - b^2a$
16. $12k^2j + 24k^2j^2$
17. $72b^2q - 18b^3q^2$
18. $4(y-3) + k(3-y)$
19. $a^2(a-1) - 25(a-1)$
20. $bm(b+4) - 6m(b+4)$
21. $a^2(a+7) + 9(a+7)$
22. $3b(b-4) - 7(4-b)$
23. $a^2b^2c^2 - 1$

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(1.-7.) 00br (8.-12.) 00bs (13.-18.) 00bt (19.-23.) 00bu

Factorising by grouping in pairs

EMAO

The taking out of common factors is the starting point in all factorisation problems. We know that the factors of $3x + 3$ are 3 and $(x + 1)$. Similarly, the factors of $2x^2 + 2x$ are $2x$ and $(x + 1)$. Therefore, if we have an expression

$$2x^2 + 2x + 3x + 3$$

there is no common factor to all four terms, but we can factorise as follows:

$$(2x^2 + 2x) + (3x + 3) = 2x(x + 1) + 3(x + 1)$$

We can see that there is another common factor $(x + 1)$. Therefore, we can now write:

$$(x + 1)(2x + 3)$$

We get this by taking out the $(x + 1)$ and seeing what is left over. We have $2x$ from the first group and $+3$ from the second group. This is called factorising by grouping.

Example 12: *Factorising by grouping in pairs*

QUESTION

Find the factors of $7x + 14y + bx + 2by$.

SOLUTION

Step 1 : ***There are no factors common to all terms***

Step 2 : ***Group terms with common factors together***

7 is a common factor of the first two terms and b is a common factor of the second two terms. We see that the ratio of the coefficients $7 : 14$ is the same as $b : 2b$.

$$\begin{aligned} 7x + 14y + bx + 2by &= (7x + 14y) + (bx + 2by) \\ &= 7(x + 2y) + b(x + 2y) \end{aligned}$$

Step 3 : ***Take out the common factor*** $(x + 2y)$

$$7(x + 2y) + b(x + 2y) = (x + 2y)(7 + b)$$

OR

Step 1 : ***Group terms with common factors together***

x is a common factor of the first and third terms and $2y$ is a common factor of the second and fourth terms

$(7 : b \;=\; 14 : 2b)$.

Step 2 : ***Rearrange the equation with grouped terms together***

$$\begin{aligned} 7x + 14y + bx + 2by &= (7x + bx) + (14y + 2by) \\ &= x(7 + b) + 2y(7 + b) \end{aligned}$$

Step 3 : ***Take out the common factor*** $(7 + b)$

$$x(7 + b) + 2y(7 + b) = (7 + b)(x + 2y)$$

Step 4 : ***Write the final answer***

The factors of $7x + 14y + bx + 2by$ are $(7 + b)$ and $(x + 2y)$.

Exercise 1 - 6

Factorise the following:

1. $6x + a + 2ax + 3$
2. $x^2 - 6x + 5x - 30$
3. $5x + 10y - ax - 2ay$
4. $a^2 - 2a - ax + 2x$
5. $5xy - 3y + 10x - 6$
6. $ab - a^2 - a + b$

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(1.) 00bz (2.) 00c0 (3.) 00c1 (4.) 00c2 (5.) 00c3

(6.) 00c4

Factorising a quadratic trinomial EMAP

Factorising is the reverse of calculating the product of factors. In order to factorise a quadratic, we need to find the factors which, when multiplied together, equal the original quadratic.

Consider a quadratic expression of the form $ax^2 + bx$. We see here that x is a common factor in both terms. Therefore, $ax^2 + bx$ factorises as $x(ax + b)$. For example, $8y^2 + 4y$ factorises as $4y(2y+1)$. Another type of quadratic is made up of the difference of squares. We know that:

$$(a + b)(a - b) = a^2 - b^2$$

So $a^2 - b^2$ can be written in factorised form as $(a + b)(a - b)$.

This means that if we ever come across a quadratic that is made up of a difference of squares, we can immediately write down the factors. These types of quadratics are very simple to factorise. However, many quadratics do not fall into these categories and we need a more general method to factorise quadratics. We can learn about factorising quadratics by looking at the opposite process, where two binomials are multiplied to get a quadratic. For example,

$$\begin{aligned}(x + 2)(x + 3) &= x^2 + 3x + 2x + 6 \\ &= x^2 + 5x + 6\end{aligned}$$

We see that the x^2 term in the quadratic is the product of the x-terms in each bracket. Similarly, the 6 in the quadratic is the product of the 2 and 3 in the brackets. Finally, the middle term is the sum of two terms.

So, how do we use this information to factorise the quadratic?

Let us start with factorising x^2+5x+6 and see if we can decide upon some general rules. Firstly, write down two brackets with an x in each bracket and space for the remaining terms.

$$(x \qquad)(x \qquad)$$

Next, decide upon the factors of 6. Since the 6 is positive, possible combinations are:

Factors of 6	
1	6
2	3
-1	-6
-2	-3

Therefore, we have four possibilities:

Option 1	**Option 2**	**Option 3**	**Option 4**
$(x+1)(x+6)$	$(x-1)(x-6)$	$(x+2)(x+3)$	$(x-2)(x-3)$

Next, we expand each set of brackets to see which option gives us the correct middle term.

Option 1	**Option 2**	**Option 3**	**Option 4**
$(x+1)(x+6)$	$(x-1)(x-6)$	$(x+2)(x+3)$	$(x-2)(x-3)$
x^2+7x+6	x^2-7x+6	x^2+5x+6	x^2-5x+6

We see that Option 3, $(x+2)(x+3)$, is the correct solution.

The process of factorising a quadratic is mostly trial and error but there are some strategies that can be used to ease the process.

Video: MG10064 at www.everythingmaths.co.za

General procedure for factorising a trinomial EMAQ

1. Divide the entire equation by any common factor of the coefficients so as to obtain an equation of the form ax^2+bx+c where a, b and c have no common factors and a is positive.
2. Write down two brackets with an x in each bracket and space for the remaining terms:
$$(\ x \qquad)(\ x \qquad)$$
3. Write down a set of factors for a and c.
4. Write down a set of options for the possible factors for the quadratic using the factors of a and c.
5. Expand all options to see which one gives you the correct middle term bx.

Note: If c is positive, then the factors of c must be either both positive or both negative. If c is negative, it means only one of the factors of c is negative, the other one being positive. Once you get an answer, always multiply out your brackets again just to make sure it really works.

Example 13: *Factorising a quadratic trinomial*

QUESTION

Factorise: $3x^2 + 2x - 1$.

SOLUTION

Step 1 : ***Check that the quadratic is in required form*** $ax^2 + bx + c$

Step 2 : ***Write down a set of factors for*** a ***and*** c

$$(\ x \qquad)(\ x \qquad)$$

The possible factors for a are: $(1;\ 3)$.
The possible factors for c are: $(-1;\ 1)$ or $(1;\ -1)$.
Write down a set of options for the possible factors of the quadratic using the factors of a and c. Therefore, there are two possible options.

Option 1	**Option 2**
$(x-1)(3x+1)$	$(x+1)(3x-1)$
$3x^2 - 2x - 1$	$3x^2 + 2x - 1$

Step 3 : ***Check that the solution is correct by multiplying the factors***

$$\begin{aligned}(x+1)(3x-1) &= 3x^2 - x + 3x - 1\\ &= 3x^2 + 2x - 1\end{aligned}$$

Step 4 : ***Write the final answer***

The factors of $3x^2 + 2x - 1$ are $(x + 1)$ and $(3x - 1)$.

Exercise 1 - 7

1. Factorise the following:

(a) $x^2 + 8x + 15$

(b) $x^2 + 10x + 24$

(c) $x^2 + 9x + 8$

(d) $x^2 + 9x + 14$

(e) $x^2 + 15x + 36$

(f) $x^2 + 12x + 36$

2. Write the following expressions in factorised form:

(a) $x^2 - 2x - 15$

(b) $x^2 + 2x - 3$

(c) $x^2 + 2x - 8$

(d) $x^2 + x - 20$

(e) $x^2 - x - 20$

(f) $2x^2 + 22x + 20$

3. Find the factors of the following trinomial expressions:

(a) $3x^2 + 19x + 6$

(b) $6x^2 + 7x + 1$

(c) $12x^2 + 8x + 1$

(d) $8x^2 + 6x + 1$

4. Factorise:

(a) $3x^2 + 17x - 6$

(b) $7x^2 - 6x - 1$

(c) $8x^2 - 6x + 1$

(d) $6x^2 - 15x - 9$

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(1.) 00bv (2.) 00bw (3.) 00bx (4.) 00by

Sum and difference of two cubes

EMAR

We now look at two special results obtained from multiplying a binomial and a trinomial:

Sum of two cubes:

$$
\begin{aligned}
(x+y)(x^2-xy+y^2) &= x(x^2-xy+y^2)+y(x^2-xy+y^2)\\
&= \left[x(x^2)+x(-xy)+x(y^2)\right]+\left[y(x^2)+y(-xy)+y(y^2)\right]\\
&= x^3-x^2y+xy^2+x^2y-xy^2+y^3\\
&= x^3+y^3
\end{aligned}
$$

Difference of two cubes:

$$
\begin{aligned}
(x-y)(x^2+xy+y^2) &= x(x^2+xy+y^2)-y(x^2+xy+y^2)\\
&= \left[x(x^2)+x(xy)+x(y^2)\right]-\left[y(x^2)+y(xy)+y(y^2)\right]\\
&= x^3+x^2y+xy^2-x^2y-xy^2-y^3\\
&= x^3-y^3
\end{aligned}
$$

So we have seen that:

$$
\begin{aligned}
x^3+y^3 &= (x+y)(x^2-xy+y^2)\\
x^3-y^3 &= (x-y)(x^2+xy+y^2)
\end{aligned}
$$

We use these two basic equations to factorise more complex examples.

Example 14: *Factorising a difference of two cubes*

QUESTION

Factorise: $x^3 - 1$.

SOLUTION

Step 1 : ***Take the cube root of terms that are perfect cubes***

Notice that $\sqrt[3]{x^3} = x$ and $\sqrt[3]{1} = 1$. These give the terms in the first bracket.

Step 2 : ***Use inspection to find the three terms in the second bracket***

$$(x^3 - 1) = (x - 1)(x^2 + x + 1)$$

Step 3 : ***Expand the brackets to check that the expression has been correctly factorised***

$$\begin{aligned}(x - 1)(x^2 + x + 1) &= x(x^2 + x + 1) - 1(x^2 + x + 1)\\ &= x^3 + x^2 + x - x^2 - x - 1\\ &= x^3 - 1\end{aligned}$$

Example 15: *Factorising a sum of two cubes*

QUESTION

Factorise: $x^3 + 8$.

SOLUTION

Step 1 : ***Take the cube root of terms that are perfect cubes***

Notice that $\sqrt[3]{x^3} = x$ and $\sqrt[3]{8} = 2$. These give the terms in the first bracket.

Step 2 : ***Use inspection to find the three terms in the second bracket***

$$(x^3 + 8) = (x + 2)(x^2 - 2x + 4)$$

Step 3 : ***Expand the brackets to check that the expression has been correctly factorised***

$$\begin{aligned}(x + 2)(x^2 - 2x + 4) &= x(x^2 - 2x + 4) + 2(x^2 - 2x + 4)\\ &= x^3 - 2x^2 + 4x + 2x^2 - 4x + 8\\ &= x^3 + 8\end{aligned}$$

Example 16: *Factorising a difference of two cubes*

QUESTION

Factorise: $16y^3 - 432$.

SOLUTION

Step 1 : ***Take out the common factor*** 16

$$16y^3 - 432 = 16(y^3 - 27)$$

Step 2 : ***Take the cube root of terms that are perfect cubes***

Notice that $\sqrt[3]{y^3} = y$ and $\sqrt[3]{27} = 3$. These give the terms in the first bracket.

Step 3 : ***Use inspection to find the three terms in the second bracket***

$$16(y^3 - 27) = 16(y - 3)(y^2 + 3y + 9)$$

Step 4 : ***Expand the brackets to check that the expression has been correctly factorised***

$$\begin{aligned}16(y-3)(y^2+3y+9) &= 16[(y(y^2+3y+9)-3(y^2+3y+9)]\\ &= 16[y^3+3y^2+9y-3y^2-9y-27]\\ &= 16y^3-432\end{aligned}$$

Example 17: *Factorising a sum of two cubes*

QUESTION

Factorise: $8t^3+125p^3$.

SOLUTION

Step 1 : ***There is no common factor***

Step 2 : ***Take the cube roots***

Notice that $\sqrt[3]{8t^3}=2t$ and $\sqrt[3]{125p^3}=5p$. These give the terms in the first bracket.

Step 3 : ***Use inspection to find the three terms in second bracket***

$$\begin{aligned}(8t^3+125p^3) &= (2t+5p)\left[(2t)^2-(2t)(5p)+(5p)^2\right]\\ &= (2t+5p)(4t^2-10tp+25p^2)\end{aligned}$$

Step 4 : ***Expand the brackets to check that expression has been correctly factorised***

$$(2t+5p)(4t^2-10tp+25p^2) = 2t(4t^2-10tp+25p^2)+5p(4t^2-10tp+25p^2)$$

$$= 8t^3 - 20pt^2 + 50p^2t + 20pt^2 - 50p^2t + 125p^3$$
$$= 8t^3 + 125p^3$$

Exercise 1 - 8

Factorise:

1. $x^3 + 8$
2. $27 - m^3$
3. $2x^3 - 2y^3$
4. $3k^3 + 27q^3$
5. $64t^3 - 1$
6. $64x^2 - 1$
7. $125x^3 + 1$
8. $25x^2 + 1$
9. $z - 125z^4$
10. $8m^6 + n^9$
11. $p^{15} - \frac{1}{8}y^{12}$
12. $1 - (x - y)^3$

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(1.-5.) 00c5 (6.-10.) 00c6 (11.-12.) 022k

1.7 Simplification of fractions

EMAS

We have studied procedures for working with fractions in earlier grades.

1. $\frac{a}{b} \times \frac{c}{d} = \frac{ac}{bd} \qquad (b \neq 0;\ d \neq 0)$
2. $\frac{a}{b} + \frac{c}{b} = \frac{a+c}{b} \qquad (b \neq 0)$
3. $\frac{a}{b} \div \frac{c}{d} = \frac{a}{b} \times \frac{d}{c} = \frac{ad}{bc} \qquad (b \neq 0;\ c \neq 0;\ d \neq 0)$

Note: dividing by a fraction is the same as multiplying by the reciprocal of the fraction.

In some cases of simplifying an algebraic expression, the expression will be a fraction. For example,

$$\frac{x^2+3x}{x+3}$$

has a quadratic binomial in the numerator and a linear binomial in the denominator. We have to apply the different factorisation methods in order to factorise the numerator and the denominator before we can simplify the expression.

$$\begin{aligned}\frac{x^2+3x}{x+3} &= \frac{x(x+3)}{x+3} \\ &= x \qquad\qquad (x \neq -3)\end{aligned}$$

If $x = -3$ then the denominator, $x + 3 = 0$, and the fraction is undefined.

Video: VMacc at www.everythingmaths.co.za

Example 18: *Simplifying fractions*

QUESTION

Simplify: $\frac{ax-b+x-ab}{ax^2-abx}$, $(x \neq 0; x \neq b)$.

SOLUTION

Step 1 : ***Use grouping to factorise the numerator and take out the common factor*** ax ***in the denominator***

$$\frac{(ax-ab)+(x-b)}{ax^2-abx} = \frac{a(x-b)+(x-b)}{ax(x-b)}$$

Step 2 : ***Take out common factor*** $(x-b)$ ***in the numerator***

$$= \frac{(x-b)(a+1)}{ax(x-b)}$$

Step 3 : ***Cancel the common factor in the numerator and the denominator to give the final answer***

$$= \frac{a+1}{ax}$$

Example 19: *Simplifying fractions*

QUESTION

Simplify: $\dfrac{x^2-x-2}{x^2-4} \div \dfrac{x^2+x}{x^2+2x}, \quad (x \neq 0; x \neq \pm 2).$

SOLUTION

Step 1 : ***Factorise the numerator and denominator***

$$= \frac{(x+1)(x-2)}{(x+2)(x-2)} \div \frac{x(x+1)}{x(x+2)}$$

Step 2 : ***Change the division sign and multiply by the reciprocal***

$$= \frac{(x+1)(x-2)}{(x+2)(x-2)} \times \frac{x(x+2)}{x(x+1)}$$

Step 3 : ***Write the final answer***

$$= 1$$

Example 20: *Simplifying fractions*

QUESTION

Simplify: $\dfrac{x-2}{x^2-4} + \dfrac{x^2}{x-2} - \dfrac{x^3+x-4}{x^2-4}, \quad (x \neq \pm 2).$

SOLUTION

Step 1 : ***Factorise the denominators***

$$\frac{x-2}{(x+2)(x-2)} + \frac{x^2}{x-2} - \frac{x^3+x-4}{(x+2)(x-2)}$$

Step 2 : ***Make all denominators the same so that we can add or subtract the fractions***

The lowest common denominator is $(x-2)(x+2)$.

$$\frac{x-2}{(x+2)(x-2)} + \frac{(x^2)(x+2)}{(x+2)(x-2)} - \frac{x^3+x-4}{(x+2)(x-2)}$$

Step 3 : ***Write as one fraction***

$$\frac{x-2+(x^2)(x+2)-(x^3+x-4)}{(x+2)(x-2)}$$

Step 4 : ***Simplify***

$$\frac{x-2+x^3+2x^2-x^3-x+4}{(x+2)(x-2)} = \frac{2x^2+2}{(x+2)(x-2)}$$

Step 5 : ***Take out the common factor and write the final answer***

$$\frac{2(x^2+1)}{(x+2)(x-2)}$$

Example 21: *Simplifying fractions*

QUESTION

Simplify: $\frac{2}{x^2-x}+\frac{x^2+x+1}{x^3-1}-\frac{x}{x^2-1}, \quad (x \neq 0; x \neq \pm 1).$

SOLUTION

Step 1 : ***Factorise the numerator and denominator***

$$\frac{2}{x(x-1)}+\frac{(x^2+x+1)}{(x-1)(x^2+x+1)}-\frac{x}{(x-1)(x+1)}$$

Step 2 : ***Simplify and find the common denominator***

$$\frac{2(x+1)+x(x+1)-x^2}{x(x-1)(x+1)}$$

Step 3 : ***Write the final answer***

$$\frac{2x+2+x^2+x-x^2}{x(x-1)(x+1)}=\frac{3x+2}{x(x-1)(x+1)}$$

Exercise 1 - 9

Simplify (assume all denominators are non-zero):

1. $\frac{3a}{15}$
2. $\frac{2a+10}{4}$
3. $\frac{5a+20}{a+4}$
4. $\frac{a^2-4a}{a-4}$
5. $\frac{3a^2-9a}{2a-6}$
6. $\frac{9a+27}{9a+18}$
7. $\frac{6ab+2a}{2b}$
8. $\frac{16x^2y-8xy}{12x-6}$
9. $\frac{4xyp-8xp}{12xy}$
10. $\frac{3a+9}{14}\div\frac{7a+21}{a+3}$

11. $\frac{a^2 - 5a}{2a + 10} \times \frac{4a}{3a + 15}$
12. $\frac{3xp + 4p}{8p} \div \frac{12p^2}{3x + 4}$
13. $\frac{24a - 8}{12} \div \frac{9a - 3}{6}$
14. $\frac{a^2 + 2a}{5} \div \frac{2a + 4}{20}$
15. $\frac{p^2 + pq}{7p} \times \frac{21q}{8p + 8q}$
16. $\frac{5ab - 15b}{4a - 12} \div \frac{6b^2}{a + b}$
17. $\frac{f^2a - fa^2}{f - a}$
18. $\frac{2}{xy} + \frac{4}{xz} + \frac{3}{yz}$
19. $\frac{5}{t - 2} - \frac{1}{t - 3}$
20. $\frac{k + 2}{k^2 + 2} - \frac{1}{k + 2}$
21. $\frac{t + 2}{3q} + \frac{t + 1}{2q}$
22. $\frac{3}{p^2 - 4} + \frac{2}{(p - 2)^2}$
23. $\frac{x}{x + y} + \frac{x^2}{y^2 - x^2}$
24. $\frac{1}{m + n} + \frac{3mn}{m^3 + n^3}$
25. $\frac{h}{h^3 - f^3} - \frac{1}{h^2 + hf + f^2}$
26. $\frac{x^2 - 1}{3} \times \frac{1}{x - 1} - \frac{1}{2}$
27. $\frac{x^2 - 2x + 1}{(x - 1)^3} - \frac{x^2 + x + 1}{x^3 - 1}$
28. $\frac{1}{(x - 1)^2} - \frac{2x}{x^3 - 1}$
29. $\frac{p^3 + q^3}{p^2} \times \frac{3p - 3q}{p^2 - q^2}$
30. $\frac{1}{a^2 - 4ab + 4b^2} + \frac{a^2 + 2ab + b^2}{a^3 - 8b^3} - \frac{1}{a^2 - 4b^2}$

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(1.-7.) 00c7 (8.-13.) 00c8 (14.-20.) 00c9 (21.-26.) 00ca (27.-30.) 022m

Chapter 1 | Summary

Summary presentation: VMcyl at www.everythingmaths.co.za

- A rational number is any number that can be written as $\frac{a}{b}$ where a and b are integers and $b \neq 0$.
- The following are rational numbers:
  - Fractions with both numerator and denominator as integers
  - Integers

  - Decimal numbers that terminate
  - Decimal numbers that recur (repeat)

- Irrational numbers are numbers that cannot be written as a fraction with the numerator and denominator as integers.

- If the n^{th} root of a number cannot be simplified to a rational number, it is called a surd.

- If a and b are positive whole numbers, and $a < b$, then $\sqrt[n]{a} < \sqrt[n]{b}$.

- A binomial is an expression with two terms.

- The product of two identical binomials is known as the square of the binomial.

- We get the difference of two squares when we multiply $(ax + b)(ax - b)$.

- Factorising is the opposite process of expanding the brackets.

- The product of a binomial and a trinomial is:

$$(A + B)(C + D + E) = A(C + D + E) + B(C + D + E)$$

- Taking out a common factor is the basic factorisation method.

- We often need to use grouping to factorise polynomials.

- To factorise a quadratic we find the two binomials that were multiplied together to give the quadratic.

- The sum of two cubes can be factorised as: $x^3 + y^3 = (x + y)(x^2 - xy + y^2)$.

- The difference of two cubes can be factorised as: $x^3 - y^3 = (x - y)(x^2 + xy + y^2)$.

- We can simplify fractions by incorporating the methods we have learnt to factorise expressions.

- Only factors can be cancelled out in fractions, never terms.

- To add or subtract fractions, the denominators of all the fractions must be the same.

Chapter 1 End of Chapter Exercises

1. If a is an integer, b is an integer and c is irrational, which of the following are rational numbers?
   (a) $\frac{-b}{a}$
   (b) $c \div c$
   (c) $\frac{a}{c}$
   (d) $\frac{1}{c}$
2. Write each decimal as a simple fraction.
   (a) $0{,}12$
   (b) $0{,}006$
   (c) $1{,}59$
   (d) $12{,}27\dot{7}$
3. Show that the decimal $3{,}21\dot{1}\dot{8}$ is a rational number.
4. Express $0{,}7\dot{8}$ as a fraction $\frac{a}{b}$ where $a,b \in \mathbb{Z}$ (show all working).
5. Write the following rational numbers to 2 decimal places.
   (a) $\frac{1}{2}$
   (b) 1
   (c) $0{,}11111\overline{1}$
   (d) $0{,}99999\overline{1}$
6. Round off the following irrational numbers to 3 decimal places.
   (a) $3{,}141592654\ldots$
   (b) $1{,}618033989\ldots$
   (c) $1{,}41421356\ldots$
   (d) $2{,}71828182845904523536\ldots$
7. Use your calculator and write the following irrational numbers to 3 decimal places.
   (a) $\sqrt{2}$
   (b) $\sqrt{3}$
   (c) $\sqrt{5}$
   (d) $\sqrt{6}$
8. Use your calculator (where necessary) and write the following numbers to 5 decimal places. State whether the numbers are irrational or rational.

(a) $\sqrt{8}$

(b) $\sqrt{768}$

(c) $\sqrt{0{,}49}$

(d) $\sqrt{0{,}0016}$

(e) $\sqrt{0{,}25}$

(f) $\sqrt{36}$

(g) $\sqrt{1960}$

(h) $\sqrt{0{,}0036}$

(i) $-8\sqrt{0{,}04}$

(j) $5\sqrt{80}$

9. Write the following irrational numbers to 3 decimal places and then write each one as a rational number to get an approximation to the irrational number.

(a) $3{,}141592654\ldots$

(b) $1{,}618033989\ldots$

(c) $1{,}41421356\ldots$

(d) $2{,}718281828459045235 36\ldots$

10. Determine between which two consecutive integers the following irrational numbers lie, without using a calculator.

(a) $\sqrt{5}$

(b) $\sqrt{10}$

(c) $\sqrt{20}$

(d) $\sqrt{30}$

(e) $\sqrt[3]{5}$

(f) $\sqrt[3]{10}$

(g) $\sqrt[3]{20}$

(h) $\sqrt[3]{30}$

11. Find two consecutive integers such that $\sqrt{7}$ lies between them.

12. Find two consecutive integers such that $\sqrt{15}$ lies between them.

13. Factorise:

(a) $a^2 - 9$

(b) $m^2 - 36$

(c) $9b^2 - 81$

(d) $16b^6 - 25a^2$

(e) $m^2 - \frac{1}{9}$

(f) $5 - 5a^2b^6$

(g) $16ba^4 - 81b$

(h) $a^2 - 10a + 25$

(i) $16b^2 + 56b + 49$

(j) $2a^2 - 12ab + 18b^2$

(k) $-4b^2 - 144b^8 + 48b^5$

(l) $(16 - x^4)$

(m) $7x^2 - 14x + 7xy - 14y$

(n) $y^2 - 7y - 30$

(o) $1 - x - x^2 + x^3$

(p) $-3(1 - p^2) + p + 1$

(q) $x - x^3 + y - y^3$

(r) $x^2 - 2x + 1 - y^4$

(s) $4b(x^3 - 1) + x(1 - x^3)$

(t) $3p^3 - \frac{1}{9}$

(u) $8x^6 - 125y^9$

(v) $(2 + p)^3 - 8(p + 1)^3$

14. Simplify the following:
    (a) $(a-2)^2 - a(a+4)$
    (b) $(5a-4b)(25a^2+20ab+16b^2)$
    (c) $(2m-3)(4m^2+9)(2m+3)$
    (d) $(a+2b-c)(a+2b+c)$
    (e) $\dfrac{p^2-q^2}{p} \div \dfrac{p+q}{p^2-pq}$
    (f) $\dfrac{2}{x}+\dfrac{x}{2}-\dfrac{2x}{3}$
    (g) $\dfrac{1}{a+7}-\dfrac{a+7}{a^2-49}$
    (h) $\dfrac{x+2}{2x^3}+16$
    (i) $\dfrac{1-2a}{4a^2-1}-\dfrac{a-1}{2a^2-3a+1}-\dfrac{1}{1-a}$
    (j) $\dfrac{x^2+2x}{x^2+x+6}\times\dfrac{x^2+2x+1}{x^2+3x+2}$
15. Show that $(2x-1)^2-(x-3)^2$ can be simplified to $(x+2)(3x-4)$.
16. What must be added to x^2-x+4 to make it equal to $(x+2)^2$?
17. Evaluate $\dfrac{x^3+1}{x^2-x+1}$ if $x = 7{,}85$ without using a calculator. Show your work.
18. With what expression must $(a-2b)$ be multiplied to get a product of a^3-8b^3?
19. With what expression must $27x^3+1$ be divided to get a quotient of $3x+1$?

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(1.) 00cb	(2.) 00cc	(3.) 00cd	(4.) 00ce	(5.) 00cf
(6.) 00cg	(7.) 00ch	(8.) 00ci	(9.) 00cj	(10.) 00ck
(11.) 00cm	(12.) 00cn	(13a-k.) 00cp	(13l-p.) 00cq	(13q-t.) 00cr
(13u-v.) 022n	(14a-d.) 00cs	(14e-f.) 00ct	(14g-j.) 00cu	(15.) 00cv
(16.) 00cw	(17.) 00cx	(18.) 022p	(19.) 022q	

Equations and inequalities 2

2.1 Solving linear equations EMAT

The simplest equation to solve is a linear equation. A linear equation is an equation where the highest exponent of the variable is 1. The following are examples of linear equations:

$$2x + 2 = 1$$
$$\frac{2 - x}{3x + 1} = 2$$
$$4(2x - 9) - 4x = 4 - 6x$$
$$\frac{2a - 3}{3} - 3a = \frac{a}{3}$$

Solving an equation means finding the value of the variable that makes the equation true. For example, to solve the simple equation $x + 1 = 1$, we need to determine the value of x that will make the left hand side equal to the right hand side. The solution is $x = 0$.

The solution, also called the root of an equation, is the value of the variable that satisfies the equation. For linear equations, there is at most one solution for the equation.

To solve equations we use algebraic methods that include expanding expressions, grouping terms, and factorising.
For example,

$$\begin{aligned} 2x + 2 &= 1 \\ 2x &= 1 - 2 && \text{(rearrange)} \\ 2x &= -1 && \text{(simplify)} \\ x &= -\tfrac{1}{2} && \text{(divide both sides by 2)} \end{aligned}$$

Check the answer by substituting $x = -\frac{1}{2}$ back into the original equation.

$$\begin{aligned}
\text{LHS} &= 2x + 2 \\
&= 2(-\frac{1}{2}) + 2 \\
&= -1 + 2 \\
&= 1 \\
\text{RHS} &= 1
\end{aligned}$$

Therefore $x = -\frac{1}{2}$ is a valid solution.

Video: VMacw at www.everythingmaths.co.za

Method for solving linear equations *EMAU*

The general steps for solving linear equations are:

1. Expand all brackets.
2. Rearrange the terms so that all terms containing the variable are on one side of the equation and all constant terms are on the other side.
3. Group like terms together and simplify.
4. Factorise if necessary.
5. Find the solution and write down the answer.
6. Check the answer by substituting the solution back into the original equation.

Remember: an equation must always be balanced, whatever you do to the left-hand side, you must also do to the right-hand side.

Example 1: *Solving linear equations*

QUESTION

Solve for x: $4(2x-9)-4x=4-6x$.

SOLUTION

Step 1 : ***Expand the brackets and simplify***

$$\begin{aligned} 4(2x-9)-4x &= 4-6x \\ 8x-36-4x &= 4-6x \\ 8x-4x+6x &= 4+36 \\ 10x &= 40 \end{aligned}$$

Step 2 : ***Divide both sides by*** 10

$$x=4$$

Step 3 : ***Check the answer by substituting the solution back into the original equation***

$$\begin{aligned} \text{LHS} &= 4\left[2(4)-9\right]-4(4) \\ &= 4(8-9)-16 \\ &= 4(-1)-16 \\ &= -4-16 \\ &= -20 \\ \text{RHS} &= 4-6(4) \\ &= 4-24 \\ &= -20 \\ \therefore \text{LHS} &= \text{RHS} \end{aligned}$$

Since both sides are equal, the answer is correct.

Example 2: *Solving linear equations*

QUESTION

Solve for x: $\frac{2-x}{3x+1} = 2$.

SOLUTION

Step 1 : ***Multiply both sides of the equation by*** $(3x+1)$

Division by 0 is not permitted so there must be a restriction $(x \neq -\frac{1}{3})$.

$$\begin{aligned}
\frac{2-x}{3x+1} &= 2 \\
(2-x) &= 2(3x+1)
\end{aligned}$$

Step 2 : ***Expand the brackets and simplify***

$$\begin{aligned}
2-x &= 6x+2 \\
-x-6x &= 2-2 \\
-7x &= 0
\end{aligned}$$

Step 3 : ***Divide both sides by*** -7

$$\begin{aligned}
x &= \frac{0}{-7} \\
x &= 0
\end{aligned}$$

Step 4 : ***Check the answer by substituting the solution back into the original equation***

$$\text{LHS} = \frac{2-(0)}{3(0)+1}$$
$$= 2$$
$$= \text{RHS}$$

Since both sides are equal, the answer is correct.

Example 3: *Solving linear equations*

QUESTION

Solve for a: $\frac{2a-3}{3} - 3a = \frac{a}{3}$.

SOLUTION

Step 1 : ***Multiply the equation by the common denominator*** 3 ***and simplify***

$$2a - 3 - 9a = a$$
$$-7a - 3 = a$$

Step 2 : ***Rearrange the terms and simplify***

$$-7a - a = 3$$
$$-8a = 3$$

Step 3 : ***Divide both sides by*** -8

$$a = -\frac{3}{8}$$

Step 4 : ***Check the answer by substituting the solution back into the original equation***

$$
\begin{aligned}
\text{LHS} &= \frac{2(-\frac{3}{8})-3}{3} - 3(-\frac{3}{8}) \\
&= \frac{(-\frac{3}{4})-\frac{12}{4}}{3} + \frac{9}{8} \\
&= \left[-\frac{15}{4} \times \frac{1}{3}\right] + \frac{9}{8} \\
&= -\frac{5}{4} + \frac{9}{8} \\
&= -\frac{10}{8} + \frac{9}{8} \\
&= -\frac{1}{8} \\
\text{RHS} &= \frac{-\frac{3}{8}}{3} \\
&= \frac{-\frac{3}{8}}{3} \\
&= -\frac{3}{8} \times \frac{1}{3} \\
&= -\frac{1}{8}
\end{aligned}
$$

$\therefore$ LHS = RHS

Since both sides are equal, the answer is correct.

Exercise 2 - 1

Solve the following equations (assume all denominators are non-zero):

1. $2y - 3 = 7$
2. $-3y = 0$
3. $16y + 4 = -10$
4. $12y + 0 = 144$
5. $7 + 5y = 62$
6. $55 = 5x + \frac{3}{4}$
7. $5x = 2x + 45$
8. $23x - 12 = 6 + 3x$
9. $12 - 6x + 34x = 2x - 24 - 64$
10. $6x + 3x = 4 - 5(2x - 3)$
11. $18 - 2p = p + 9$

12. $\frac{4}{p} = \frac{16}{24}$
13. $-(-16 - p) = 13p - 1$
14. $3f - 10 = 10$
15. $3f + 16 = 4f - 10$
16. $10f + 5 = -2f - 3f + 80$
17. $8(f - 4) = 5(f - 4)$
18. $6 = 6(f + 7) + 5f$
19. $(a - 1)^2 - 2a = (a + 3)(a - 2) - 3$
20. $-7x = x + 8(1 - x)$
21. $5 - \frac{7}{b} = \frac{2(b + 4)}{b}$
22. $\frac{x + 2}{4} - \frac{x - 6}{3} = \frac{1}{2}$
23. $3 - \frac{y - 2}{4} = 4$
24. $\frac{a + 1}{a + 2} = \frac{a - 3}{a + 1}$
25. $(x - 3)(x + 2) = x(x - 4)$
26. $1{,}5x + 3{,}125 = 1{,}25x$
27. $\frac{1}{3}P + \frac{1}{2}P - 10 = 0$
28. $1\frac{1}{4}(x - 1) - 1\frac{1}{2}(3x + 2) = 0$
29. $\frac{5}{2a} + \frac{1}{6a} - \frac{3}{a} = 2$
30. $\frac{3}{2x^2} + \frac{4}{3x} - \frac{5}{6x} = 0$

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(1.-5.) 00dn (6.-10.) 00dp (11.-15.) 00dq (16.-20.) 00dr (21.-24.) 00md (25.-30.) 022t

2.2 Solving quadratic equations

EMAV

A quadratic equation is an equation where the exponent of the variable is at most 2. The following are examples of quadratic equations:

$$2x^2 + 2x = 1$$
$$3x^2 + 2x - 1 = 0$$
$$0 = -2x^2 + 4x - 2$$

Quadratic equations differ from linear equations in that a linear equation has only one solution, while a quadratic equation has at most two solutions. There are some special situations, however, in which a quadratic equation has either one solution or no solutions.

We solve quadratic equations using factorisation. For example, in order to solve $2x^2 -$

$x - 3 = 0$, we need to write it in its equivalent factorised form as $(x + 1)(2x - 3) = 0$. Note that if $a \times b = 0$ then $a = 0$ or $b = 0$.

Video: VMaec at www.everythingmaths.co.za

Method for solving quadratic equations EMAW

1. Rewrite the equation in the required form, $ax^2 + bx + c = 0$.
2. Divide the entire equation by any common factor of the coefficients to obtain an equation of the form $ax^2 + bx + c = 0$, where a, b and c have no common factors. For example, $2x^2 + 4x + 2 = 0$ can be written as $x^2 + 2x + 1 = 0$, by dividing by 2.
3. Factorise $ax^2 + bx + c = 0$ to be of the form $(rx + s)(ux + v) = 0$.
4. The two solutions are $(rx + s) = 0$ or $(ux + v) = 0$, so $x = -\frac{s}{r}$ or $x = -\frac{v}{u}$, respectively.
5. Check the answer by substituting it back into the original equation.

Example 4: *Solving quadratic equations*

QUESTION

Solve for x: $3x^2 + 2x - 1 = 0$.

SOLUTION

Step 1 : ***The equation is already in the required form,*** $ax^2 + bx + c = 0$

Step 2 : ***Factorise***

$$(x + 1)(3x - 1) = 0$$

Step 3 : ***Solve for both factors***

We have

$$x + 1 = 0$$
$$\therefore x = -1$$

OR

$$3x - 1 = 0$$
$$\therefore x = \frac{1}{3}$$

Step 4 : ***Check both answers by substituting back into the original equation***

Step 5 : ***Write the final answer***

The solution to $3x^2 + 2x - 1 = 0$ is $x = -1$ or $x = \frac{1}{3}$.

Example 5: *Solving quadratic equations*

QUESTION

Find the roots: $0 = -2x^2 + 4x - 2$.

SOLUTION

Step 1 : ***Divide the equation by common factor*** -2

$$-2x^2 + 4x - 2 = 0$$
$$x^2 - 2x + 1 = 0$$

Step 2 : ***The equation is already in the required form,*** $ax^2 + bx + c = 0$

Step 3 : ***Factorise***

$$(x-1)(x-1) = 0$$
$$(x-1)^2 = 0$$

Step 4 : ***The quadratic is a perfect square***

This is an example of a special situation in which there is only one solution to the quadratic equation because both factors are the same.

$$x - 1 = 0$$
$$\therefore x = 1$$

Step 5 : ***Check the answer by substituting back into the original equation***

Step 6 : ***Write final answer***

The solution to $0 = -2x^2 + 4x - 2$ is $x = 1$.

Exercise 2 - 2

Solve the following equations:

1. $9x^2 - 6x - 8 = 0$

2. $5x^2 - 21x - 54 = 0$

3. $4y^2 - 9 = 0$

4. $4x^2 - 16x - 9 = 0$

5. $4x^2 - 12x = -9$

6. $20m + 25m^2 = 0$

7. $2x^2 - 5x - 12 = 0$

8. $-75x^2 + 290x = 240$

9. $2x = \frac{1}{3}x^2 - 3x + 14\frac{2}{3}$

10. $x^2 - 4x = -4$

11. $-x^2 + 4x - 6 = 4x^2 - 14x + 3$

12. $t^2 = 3t$

13. $x^2 - 10x = -25$

14. $x^2 = 18$

15. $p^2 - 6p = 7$

16. $4x^2 - 17x - 77 = 0$

17. $14x^2 + 5x = 6$

18. $2x^2 - 2x = 12$

19. $\frac{a+1}{3a-4} + \frac{9}{2a+5} + \frac{2a+3}{2a+5} = 0$

20. $\frac{3}{9a^2 - 3a + 1} - \frac{3a+4}{27a^3 + 1} = \frac{1}{9a^2 - 1}$

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(1.-5.) 00ds (6.-11.) 00dt (12.-18.) 00du (19.-20.) 022u

2.3 Solving simultaneous equations EMAX

Up to now we have solved equations with only one unknown variable. When solving for two unknown variables, two equations are required and these equations are known as simultaneous equations. The solutions are the values of the unknown variables which satisfy both equations simultaneously. In general, if there are n unknown variables, then n independent equations are required to obtain a value for each of the n variables.

An example of a system of simultaneous equations is

$$\begin{aligned} x + y &= -1 \\ 3 &= y - 2x \end{aligned}$$

We have two independent equations to solve for two unknown variables. We can solve simultaneous equations algebraically using substitution and elimination methods. We will also show that a system of simultaneous equations can be solved graphically.

Video: VMaex at www.everythingmaths.co.za

Solving by substitution EMAY

- Use the simplest of the two given equations to express one of the variables in terms of the other.
- Substitute into the second equation. By doing this we reduce the number of equations and the number of variables by one.
- We now have one equation with one unknown variable which can be solved.

- Use the solution to substitute back into the first equation to find the value of the other unknown variable.

Example 6: *Simultaneous equations*

QUESTION

Solve for x and y:

$$x - y = 1 \quad \ldots\ldots\ldots\ldots(1)$$
$$3 = y - 2x \ldots\ldots\ldots(2)$$

SOLUTION

Step 1 : ***Use equation (1) to express x in terms of y***

$$x = y + 1$$

Step 2 : ***Substitute x into equation (2) and solve for y***

$$3 = y - 2(y + 1)$$
$$3 = y - 2y - 2$$
$$5 = -y$$
$$\therefore y = -5$$

Step 3 : ***Substitute y back into equation (1) and solve for x***

$$x = (-5) + 1$$
$$\therefore x = -4$$

Step 4 : ***Check the solution by substituting the answers back into both original equations***

Step 5 : ***Write the final answer***

$$
\begin{aligned}
x &= -4 \\
y &= -5
\end{aligned}
$$

Example 7: *Simultaneous equations*

QUESTION

Solve the following system of equations:

$$
\begin{aligned}
4y + 3x &= 100 \quad \ldots\ldots\ldots\ldots (1) \\
4y - 19x &= 12 \quad \ldots\ldots\ldots\ldots (2)
\end{aligned}
$$

SOLUTION

Step 1 : ***Use either equation to express*** x ***in terms of*** y

$$
\begin{aligned}
4y + 3x &= 100 \\
3x &= 100 - 4y \\
x &= \frac{100 - 4y}{3}
\end{aligned}
$$

Step 2 : ***Substitute*** x ***into equation*** (2) ***and solve for*** y

$$
\begin{aligned}
4y - 19\left(\frac{100 - 4y}{3}\right) &= 12 \\
12y - 19(100 - 4y) &= 36
\end{aligned}
$$

$$\begin{aligned} 12y - 1\ 900 + 76y &= 36 \\ 88y &= 1\ 936 \\ \therefore y &= 22 \end{aligned}$$

Step 3 : ***Substitute*** y ***back into equation*** (1) ***and solve for*** x

$$\begin{aligned} x &= \frac{100 - 4(22)}{3} \\ &= \frac{100 - 88}{3} \\ &= \frac{12}{3} \\ \therefore x &= 4 \end{aligned}$$

Step 4 : ***Check the solution by substituting the answers back into both original equations***

Step 5 : ***Write the final answer***

$$\begin{aligned} x &= 4 \\ y &= 22 \end{aligned}$$

Solving by elimination

EMAZ

Example 8: *Simultaneous equations*

QUESTION

Solve the following system of equations:

$$3x + y = 2 \quad \ldots\ldots\ldots\ldots (1)$$
$$6x - y = 25 \quad \ldots\ldots\ldots\ldots (2)$$

SOLUTION

Step 1 : ***Make the coefficients of one of the variables the same in both equations***

The coefficients of y in the given equations are 1 and -1. Eliminate the variable y by adding equation (1) and equation (2) together:

$$\begin{array}{rrcl} & 3x + y & = & 2 \\ + & 6x - y & = & 25 \\ \hline & 9x + 0 & = & 27 \end{array}$$

Step 2 : ***Simplify and solve for x***

$$9x = 27$$
$$\therefore x = 3$$

Step 3 : ***Substitute x back into either original equation and solve for y***

$$3(3) + y = 2$$
$$y = 2 - 9$$
$$\therefore y = -7$$

Step 4 : ***Check that the solution $x = 3$ and $y = -7$ satisfies both original equations***

Step 5 : ***Write final answer***

$$x = 3$$
$$y = -7$$

Example 9: *Simultaneous equations*

QUESTION

Solve the following system of equations:

$$2a - 3b = 5 \quad \ldots\ldots\ldots\ldots(1)$$
$$3a - 2b = 20 \quad \ldots\ldots\ldots\ldots(2)$$

SOLUTION

Step 1 : ***Make the coefficients of one of the variables the same in both equations***

By multiplying equation (1) by 3 and equation (2) by 2, both coefficients of a will be 6.

$$\begin{array}{rcr} 6a - 9b & = & 15 \\ -\quad (6a - 4b & = & 40) \\ \hline 0 - 5b & = & -25 \end{array}$$

(When subtracting two equations, be careful of the signs.)

Step 2 : ***Simplify and solve for*** b

$$b = \frac{-25}{-5}$$
$$\therefore b = 5$$

Step 3 : ***Substitute value of*** b ***back into either original equation and solve for*** a

$$2a - 3(5) = 5$$
$$2a - 15 = 5$$
$$2a = 20$$
$$\therefore a = 10$$

Step 4 : ***Check that the solution*** $a = 10$ ***and*** $b = 5$ ***satisfies both original equations***

Step 5 : ***Write final answer***

$$a = 10$$
$$b = 5$$

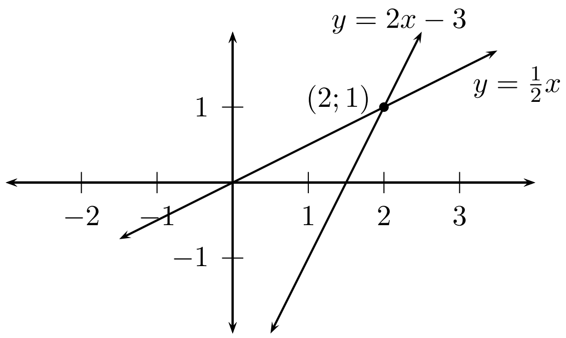
Solving graphically EMAAA

Simultaneous equations can also be solved graphically. If the graphs of each linear equation are drawn, then the solution to the system of simultaneous equations is the coordinate of the point at which the two graphs intersect.

For example:

$$x = 2y \quad \ldots\ldots\ldots\ldots\ldots (1)$$
$$y = 2x - 3 \ldots\ldots\ldots\ldots (2)$$

The graphs of the two equations are shown below.



The intersection of the two graphs is $(2; 1)$. So the solution to the system of simultaneous equations is $x = 2$ and $y = 1$. We can also check the solution using algebraic methods. Substitute equation (1) into equation (2):

$$x = 2y$$
$$\therefore y = 2(2y) - 3$$

Then solve for y:

$$y - 4y = -3$$
$$-3y = -3$$
$$\therefore y = 1$$

Substitute the value of y back into equation (1):

$$x = 2(1)$$
$$\therefore x = 2$$

Notice that both methods give the same solution.

Example 10: *Simultaneous equations*

QUESTION

Solve the following system of simultaneous equations graphically:

$$4y + 3x = 100 \ldots\ldots\ldots\ldots (1)$$
$$4y - 19x = 12 \quad \ldots\ldots\ldots\ldots (2)$$

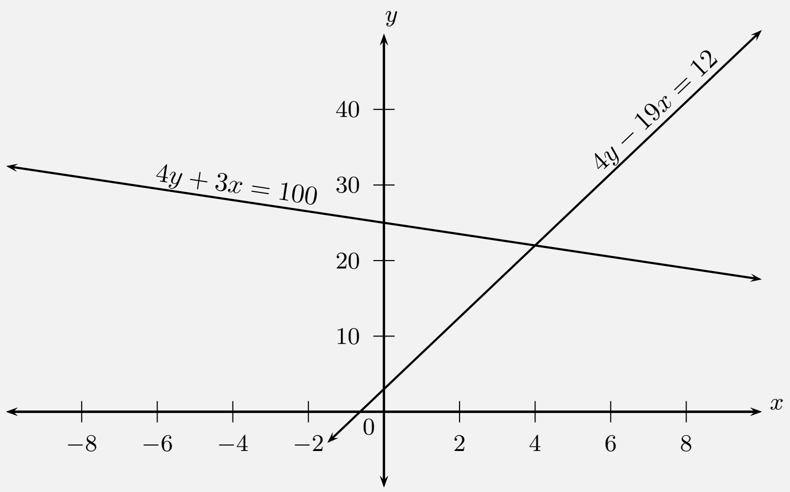
SOLUTION

Step 1 : ***Write both equations in form*** $y = mx + c$

$$\begin{aligned} 4y + 3x &= 100 \\ 4y &= 100 - 3x \\ y &= -\frac{3}{4}x + 25 \end{aligned}$$

$$\begin{aligned} 4y - 19x &= 12 \\ 4y &= 19x + 12 \\ y &= \frac{19}{4}x + 3 \end{aligned}$$

Step 2 : ***Sketch the graphs on the same set of axes***



Step 3 : ***Find the coordinates of the point of intersection***

The two graphs intersect at $(4;22)$.

Step 4 : ***Write the final answer***

$$x = 4$$
$$y = 22$$

Exercise 2 - 3

1. Solve for x and y:
   (a) $3x - 14y = 0$ and $x - 4y + 1 = 0$
   (b) $x + y = 8$ and $3x + 2y = 21$
   (c) $y = 2x + 1$ and $x + 2y + 3 = 0$
   (d) $\frac{a}{2} + b = 4$ and $\frac{a}{4} - \frac{b}{4} = 1$
   (e) $\frac{1}{x} + \frac{1}{y} = 3$ and $\frac{1}{x} - \frac{1}{y} = 11$
2. Solve graphically and check your answer algebraically:
   (a) $x + 2y = 1$ and $\frac{x}{3} + \frac{y}{2} = 1$
   (b) $5 = x + y$ and $x = y - 2$
   (c) $3x - 2y = 0$ and $x - 4y + 1 = 0$
   (d) $\frac{x}{4} = \frac{y}{2} - 1$ and $\frac{y}{4} + \frac{x}{2} = 1$
   (e) $2x + y = 5$ and $3x - 2y = 4$

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(1a-e.) 00dv (2a-e.) 00dw

2.4 Word problems

EMAAB

To solve word problems we need to write a set of equations that represent the problem mathematically. The solution of the equations is then the solution to the problem.

Problem solving strategy

EMAAC

1. Read the whole the question.
2. What are we asked to solve for?
3. Assign a variable to the unknown quantity, for example, x.
4. Translate the words into algebraic expressions by rewriting the given information in terms of the variable.
5. Set up an equation or system of equations to solve for the variable.
6. Solve the equation algebraically using substitution.
7. Check the solution.

Video: VMafb at www.everythingmaths.co.za

Example 11: *Solving word problems*

QUESTION

A shop sells bicycles and tricycles. In total there are 7 *cycles (cycles include both bicycles and tricycles) and* 19 *wheels. Determine how many of each there are, if a bicycle has two wheels and a tricycle has three wheels.*

SOLUTION

Step 1 : ***Assign variables to the unknown quantities***

Let b be the number of bicycles and let t be the number of tricycles.

Step 2 : ***Set up the equations***

$$\begin{aligned} b + t &= 7 \quad \ldots\ldots\ldots\ldots (1) \\ 2b + 3t &= 19 \ldots\ldots\ldots\ldots (2) \end{aligned}$$

Step 3 : ***Rearrange equation** (1) **and substitute into equation** (2)*

$$\begin{aligned} t &= 7 - b \\ \therefore 2b + 21 - 3b &= 19 \\ -b &= -2 \\ \therefore b &= 2 \end{aligned}$$

Step 4 : ***Calculate the number of tricycles** t*

$$\begin{aligned} t &= 7 - b \\ &= 7 - 2 \\ &= 5 \end{aligned}$$

Step 5 : ***Write the final answer***

There are 5 tricycles and 2 bicycles.

Example 12: *Solving word problems*

QUESTION

Bongani and Jane are friends. Bongani takes Jane's physics test paper and will not tell her what her mark is. He knows that Jane hates maths so he decides to tease her. Bongani says: "I have 2 marks more than you do and the sum of both our marks is equal to 14. What are our marks?"

SOLUTION

Step 1 : ***Assign variables to the unknown quantities***

We have two unknown quantities, Bongani's mark and Jane's mark. Let Bongani's mark be b and Jane's mark be j.

Step 2 : ***Set up a system of equations***

Bongani has 2 more marks than Jane.

$$b = j + 2 \quad \ldots\ldots\ldots\ldots (1)$$

Both marks add up to 14.

$$b + j = 14 \quad \ldots\ldots\ldots\ldots (2)$$

Step 3 : ***Use equation*** (1) ***to express*** b ***in terms of*** j

$$b = j + 2$$

Step 4 : ***Substitute into equation*** (2)

$$\begin{aligned} b + j &= 14 \\ (j + 2) + j &= 14 \end{aligned}$$

Step 5 : ***Rearrange and solve for*** j

$$\begin{aligned} 2j &= 14 - 2 \\ &= 12 \\ \therefore j &= 6 \end{aligned}$$

Step 6 : ***Substitute the value for*** j ***back into equation*** (1) ***and solve for*** b

$$\begin{aligned} b &= j + 2 \\ &= 6 + 2 \\ &= 8 \end{aligned}$$

Step 7 : ***Check that the solution satisfies both original equations***

Step 8 : ***Write the final answer***

Bongani got 8 for his test and Jane got 6.

Example 13: *Solving word problems*

QUESTION

A fruit shake costs R $2{,}00$ *more than a chocolate milk shake. If* 3 *fruit shakes and* 5 *chocolate milk shakes cost R* $78{,}00$*, determine the individual prices.*

SOLUTION

Step 1 : ***Assign variables to the unknown quantities***

Let the price of a chocolate milkshake be x and let the price of a fruitshake be y.

Step 2 : ***Set up a system of equations***

$$\begin{aligned} y &= x + 2 \ldots\ldots\ldots (1) \\ 3y + 5x &= 78 \ldots\ldots\ldots\ldots (2) \end{aligned}$$

Step 3 : ***Substitute equation** (1) **into equation** (2)*

$$3(x + 2) + 5x = 78$$

Step 4 : ***Rearrange and solve for** x*

$$\begin{aligned} 3x + 6 + 5x &= 78 \\ 8x &= 72 \\ \therefore x &= 9 \end{aligned}$$

Step 5 : ***Substitute the value of** x **back into equation** (1) **and solve for** y*

$$\begin{aligned} y &= x + 2 \\ &= 9 + 2 \\ &= 11 \end{aligned}$$

Step 6 : ***Check that the solution satisfies both original equations***

Step 7 : ***Write final answer***

One chocolate milkshake costs R 9,00 and one fruitshake costs R 11,00.

Example 14: *Solving word problems*

QUESTION

The product of two consecutive negative integers is $1\ 122$. *Find the two integers.*

SOLUTION

Step 1 : ***Assign variables to the unknown quantities***

Let the first integer be n and let the second integer be $n+1$.

Step 2 : ***Set up an equation***

$$n(n+1) = 1\ 122$$

Step 3 : ***Expand and solve for*** n

$$\begin{aligned} n^2 + n &= 1\ 122 \\ n^2 + n - 1\ 122 &= 0 \\ (n+34)(n-33) &= 0 \\ \therefore n &= -34 \\ \text{or } n &= 33 \end{aligned}$$

Step 4 : ***Find the sign of the integers***

It is given that both integers must be negative.

$$\begin{aligned} \therefore n &= -34 \\ n+1 &= -34+1 \\ &= -33 \end{aligned}$$

Step 5 : ***Write the final answer***

The two consecutive negative integers are -34 and -33.

Exercise 2 - 4

1. Two jets are flying towards each other from airports that are $1\ 200$ km apart. One jet is flying at 250 km/h and the other jet at 350 km/h. If they took off at the same time, how long will it take for the jets to pass each other?
2. Kadesh bought 20 shirts at a total cost of R 980. If the large shirts cost R 50 and the small shirts cost R 40. How many of each size did he buy?
3. The diagonal of a rectangle is 25 cm more than its width. The length of the rectangle is 17 cm more than its width. What are the dimensions of the rectangle?
4. The sum of 27 and 12 is equal to 73 more than an unknown number. Find the unknown number.
5. The two smaller angles in a right-angled triangle are in the ratio of $1 : 2$. What are the sizes of the two angles?
6. The length of a rectangle is twice the breadth. If the area is 128 cm^2, determine the length and the breadth.
7. If 4 times a number is increased by 6, the result is 15 less than the square of the number. Find the number.
8. The length of a rectangle is 2 cm more than the width of the rectangle. The perimeter of the rectangle is 20 cm. Find the length and the width of the rectangle.
9. Stephen has $1\ \ell$ of a mixture containing 69% salt. How much water must Stephen add to make the mixture 50% salt? Write your answer as a fraction of a litre.
10. The sum of two consecutive odd numbers is 20 and their difference is 2. Find the two numbers.
11. The denominator of a fraction is 1 more than the numerator. The sum of the fraction and its reciprocal is $\frac{5}{2}$. Find the fraction.
12. Masindi is 21 years older than her daughter, Mulivhu. The sum of their ages is 37. How old is Mulivhu?
13. Tshamano is now five times as old as his son Murunwa. Seven years from now, Tshamano will be three times as old as his son. Find their ages now.

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(1.) 00dz (2.) 00e0 (3.) 00e1 (4.) 00e2 (5.) 00e3
(6.) 00e4 (7.) 00e5 (8.) 00e6 (9.) 00e7 (10.) 022v
(11.) 022w (12.) 022x (13.) 02sm

2.5 Literal equations

EMAAD

A literal equation is one that has several letters or variables. Examples include the area of a circle ($A = \pi r^2$) and the formula for speed ($v = \frac{D}{t}$). In this section we solve literal equations in terms of one variable. To do this, we use the principles we have learnt about solving equations and apply them to rearranging literal equations. Solving literal equations is also known as changing the subject of the formula. Keep the following in mind when solving literal equations:

- We isolate the unknown variable by asking "what is it joined to?" and "how is it joined?" We then perform the opposite operation to both sides as a whole.
- If the unknown variable is in two or more terms, then we take it out as a common factor.
- If we have to take the square root of both sides, remember that there will be a positive and a negative answer.
- If the unknown variable is in the denominator, we multiply both sides by the lowest common denominator (LCD) and then continue to solve.

Video: VMajx at www.everythingmaths.co.za

Example 15: *Solving literal equations*

QUESTION

The area of a triangle is $A = \frac{1}{2}bh$. What is the height of the triangle in terms of the base and area?

SOLUTION

Step 1 : ***Isolate the required variable***

We are asked to isolate the height, so we must rearrange the equation with h on one side of the equals sign and the rest of the variables on the other.

$$\begin{aligned} A &= \frac{1}{2}bh \\ 2A &= bh \\ \frac{2A}{b} &= h \end{aligned}$$

Step 2 : ***Write the final answer***

The height of a triangle is given by: $h = \frac{2A}{b}$

Example 16: *Solving literal equations*

QUESTION

Given the formula $h = R \times \frac{H}{R + r^2}$, *make R the subject of the formula.*

SOLUTION

Step 1 : ***Isolate the required variable***

$$\begin{aligned} h(R + r^2) &= R \times H \\ hR + hr^2 &= HR \\ hr^2 &= HR - hR \\ hr^2 &= R(H - h) \\ \therefore R &= \frac{hr^2}{H - h} \end{aligned}$$

Exercise 2 - 5

1. Make a the subject of the formula: $s = ut + \frac{1}{2}at^2$
2. Solve for n: $pV = nRT$
3. Make x the subject of the formula: $\frac{1}{b} + \frac{2b}{x} = 2$
4. Solve for r: $V = \pi r^2 h$
5. Solve for h: $E = \frac{hc}{\lambda}$
6. Solve for h: $A = 2\pi rh + 2\pi r$
7. Make λ the subject of the formula: $t = \frac{D}{f\lambda}$
8. Solve for m: $E = mgh + \frac{1}{2}mv^2$
9. Solve for x: $x^2 + x(a + b) + ab = 0$
10. Solve for b: $c = \sqrt{a^2 + b^2}$
11. Make u the subject of the formula: $\frac{1}{v} = \frac{1}{u} + \frac{1}{w}$
12. Solve for r: $A = \pi R^2 - \pi r^2$
13. $F = \frac{9}{5}C + 32°$ is the formula for converting temperature in degrees Celsius to degrees Fahrenheit. Derive a formula for converting degrees Fahrenheit to degrees Celsius.
14. $V = \frac{4}{3}\pi r^3$ is the formula for determining the volume of a soccer ball. Express the radius in terms of the volume.

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(1.-3.) 00e8 (4.-8.) 00e9 (9.-14.) 022y

2.6 Solving linear inequalities

EMAAE

A linear inequality is similar to a linear equation in that the largest exponent of a variable is 1. The following are examples of linear inequalities.

$$\begin{aligned} &2x + 2 \leq 1 \\ &\frac{2-x}{3x+1} \geq 2 \\ &\frac{4}{3}x - 6 < 7x + 2 \end{aligned}$$

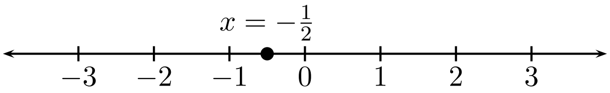
The methods used to solve linear inequalities are similar to those used to solve linear equations. The only difference occurs when there is a multiplication or a division that involves a minus sign. For example, we know that $8 > 6$. If both sides of the inequality are divided by -2, then we get $-4 > -3$, which is not true. Therefore, the inequality sign must be switched around, giving $-4 < -3$.

In order to compare an inequality to a normal equation, we shall solve an equation first.

Solve $2x + 2 = 1$:

$$\begin{aligned} 2x + 2 &= 1 \\ 2x &= 1 - 2 \\ 2x &= -1 \\ x &= -\frac{1}{2} \end{aligned}$$

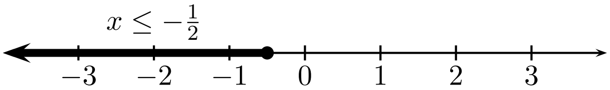
If we represent this answer on a number line, we get:



Now let us solve the inequality $2x + 2 \leq 1$:

$$\begin{aligned} 2x + 2 &\leq 1 \\ 2x &\leq 1 - 2 \\ 2x &\leq -1 \\ x &\leq -\frac{1}{2} \end{aligned}$$

If we represent this answer on a number line, we get:



We see that for the equation there is only a single value of x for which the equation is true. However, for the inequality, there is a range of values for which the inequality is true. This is the main difference between an equation and an inequality.

Remember: when we divide or multiply both sides of an inequality by any number with a minus sign, the direction of the inequality changes. For example, if $x < 1$, then $-x > -1$. Also note that we cannot divide or multiply by a variable.

Video: VMakv at www.everythingmaths.co.za

Interval notation

EMAAF

Examples:

$(4; 12)$	Round brackets indicate that the number is not included. This interval includes all real numbers greater than but not equal to 4 and less than but not equal to 12.
$(-\infty; -1)$	Round brackets are always used for positive and negative infinity. This interval includes all real numbers less than, but not equal to -1.
$[1; 13)$	A square bracket indicates that the number is included. This interval includes all real numbers greater than or equal to 1 and less than but not equal to 13.

It is important to note that this notation can only be used to represent an interval of real numbers.

We represent the above answer in interval notation as $(-\infty; \ -\frac{1}{2}]$.

Example 17: *Solving linear inequalities*

QUESTION

Solve for r: $6 - r > 2$. Represent the answer on a number line and in interval notation.

SOLUTION

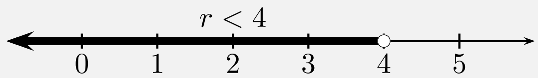
Step 1 : ***Rearrange and solve for*** r

$$-r > 2 - 6$$
$$-r > -4$$

Step 2 : ***Multiply by*** -1 ***and reverse inequality sign***

$$r < 4$$

Step 3 : ***Represent the answer on a number line***



Step 4 : ***Represent the answer in interval notation***

$$(-\infty\ ;\ 4)$$

Example 18: *Solving linear inequalities*

QUESTION

Solve for q*:* $4q + 3 < 2(q + 3)$*. Represent the answer on a number line and in interval notation.*

SOLUTION

Step 1 : ***Expand the bracket***

$$4q + 3 < 2(q + 3)$$
$$4q + 3 < 2q + 6$$

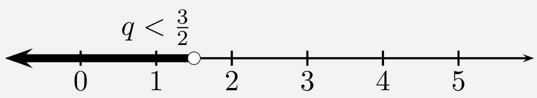
Step 2 : ***Rearrange and solve for*** q

$$4q + 3 < 2q + 6$$
$$4q - 2q < 6 - 3$$
$$2q < 3$$

Step 3 : ***Divide both sides by*** 2

$$2q < 3$$
$$q < \frac{3}{2}$$

Step 4 : ***Represent the answer on a number line***



Step 5 : ***Represent the answer in interval notation***

$$\left(-\infty\ ;\ \frac{3}{2}\right)$$

Example 19: *Solving compound linear inequalities*

QUESTION

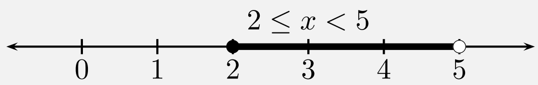
Solve for x: $5 \leq x + 3 < 8$. *Represent the answer on a number line and in interval notation.*

SOLUTION

Step 1 : ***Subtract** 3 **from all the terms***

$$\begin{aligned} 5-3 &\leq x+3-3 < 8-3 \\ 2 &\leq x < 5 \end{aligned}$$

Step 2 : ***Represent the answer on a number line***



Step 3 : ***Represent the answer in interval notation***

$$[\,2\,;\,5)$$

Exercise 2 - 6

Solve for x and represent the answer on a number line and in interval notation:

1. $3x+4 > 5x+8$

2. $3(x-1)-2 \leq 6x+4$

3. $\dfrac{x-7}{3} > \dfrac{2x-3}{2}$

4. $-4(x-1) < x+2$

5. $\dfrac{1}{2}x + \dfrac{1}{3}(x-1) \geq \dfrac{5}{6}x - \dfrac{1}{3}$

6. $-2 \leq x-1 < 3$

7. $-5 < 2x-3 \leq 7$

8. $7(3x+2)-5(2x-3) > 7$

9. $\dfrac{5x-1}{-6} \leq \dfrac{1-2x}{3}$

10. $3 \leq 4-x \leq 16$

11. $\dfrac{-7y}{3} - 5 > -7$

12. $1 \leq 1 - 2y < 9$

13. $-2 < \dfrac{x-1}{-3} < 7$

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(1.) 00ea	(2.) 00eb	(3.) 00ec	(4.) 00ed	(5.) 00ee
(6.) 00ef	(7.) 00eg	(8.) 00eh	(9.) 022z	(10.) 0230
(11.) 0231	(12.) 0232	(13.) 0233		

Chapter 2 | Summary

Summary presentation: VMjkd at www.everythingmaths.co.za

- A linear equation is an equation where the exponent of the variable is 1. A linear equation has at most one solution.
- A quadratic equation is an equation where the exponent of the variable is at most 2. A quadratic equation has at most two solutions.
- A linear inequality is similar to a linear equation and has the exponent of the variable equal to 1.
- If we divide or multiply both sides of an inequality by any number with a minus sign, the direction of the inequality changes.
- To solve for two unknown variables, two equations are required. These equations are known as a system of simultaneous equations. There are two ways to solve linear simultaneous equations: algebraic solutions and graphical solutions. To solve algebraically we use substitution or elimination methods. To solve graphically we draw the graph of each equation and the solution will be the coordinates of the point of intersection.
- Literal equations are equations that have several letters and variables.
- Word problems require a set of equations that represent the problem mathematically.

Chapter 2 End of Chapter Exercises

1. Solve:

   (a) $2(p-1) = 3(p+2)$

   (b) $3 - 6k = 2k - 1$

   (c) $m + 6(-m+1) + 5m = 0$

   (d) $2k + 3 = 2 - 3(k+3)$

   (e) $5t - 1 = t^2 - (t+2)(t-2)$

   (f) $3 + \frac{q}{5} = \frac{q}{2}$

   (g) $5 - \frac{2(m+4)}{m} = \frac{7}{m}$

   (h) $\frac{2}{t} - 2 - \frac{1}{2} = \frac{1}{2}\left(1 + \frac{2}{t}\right)$

   (i) $x^2 - 3x + 2 = 0$

   (j) $y^2 + y = 6$

   (k) $0 = 2x^2 - 5x - 18$

   (l) $(d+4)(d-3) - d$
   $= (3d-2)^2 - 8d(d-1)$

   (m) $5x + 2 \leq 4(2x-1)$

   (n) $\frac{4x-2}{6} > 2x + 1$

   (o) $\frac{x}{3} - 14 > 14 - \frac{x}{7}$

   (p) $\frac{1-a}{2} - \frac{2-a}{3} \geq 1$

   (q) $-5 \leq 2k + 1 < 5$

   (r) $x - 1 = \frac{42}{x}$

2. Consider the following literal equations:

   (a) Solve for I: $P = VI$

   (b) Make m the subject of the formula: $E = mc^2$

   (c) Solve for t: $v = u + at$

   (d) Make f the subject of the formula: $\frac{1}{u} + \frac{1}{v} = \frac{1}{f}$

   (e) Make C the subject of the formula: $F = \frac{9}{5}C + 32$

   (f) Solve for y: $m = \frac{y-c}{x}$

3. Solve the following simultaneous equations:

   (a) $7x + 3y = 13$
   $2x - 3y = -4$

   (b) $10 = 2x + y$
   $y = x - 2$

   (c) $7x - 41 = 3y$
   $17 = 3x - y$

   (d) $2y = x + 8$
   $4y = 2x - 44$

4. Find the solutions to the following word problems:

(a) $\frac{7}{8}$ of a certain number is 5 more than of $\frac{1}{3}$ of the number. Find the number.

(b) Three rulers and two pens have a total cost of R $21{,}00$. One ruler and one pen have a total cost of R $8{,}00$. How much does a ruler cost and how much does a pen cost?

(c) A man runs to the bus stop and back in 15 minutes. His speed on the way to the bus stop is 5 km/h and his speed on the way back is 4 km/h. Find the distance to the bus stop.

(d) Zanele and Piet skate towards each other on a straight path. They set off 20 km apart. Zanele skates at 15 km/h and Piet at 10 km/h. How far will Piet have skated when they reach each other?

(e) When the price of chocolates is increased by R 10, we can buy five fewer chocolates for R 300. What was the price of each chocolate before the price was increased?

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(1a-f.) 00ei (1g-l.) 00ej (1m-r.) 00ek (2a-f.) 00em (3a-d.) 00en

(4a-e.) 00ep

Exponents 3

Exponential notation is a short way of writing the same number multiplied by itself many times. We will now have a closer look at writing numbers using exponential notation. Exponents can also be called indices.

$$\text{base} \longleftarrow a^n \longrightarrow \text{exponent/index}$$

For any real number a and natural number n, we can write a multiplied by itself n times as a^n.

1. $a^n = a \times a \times a \times \cdots \times a$ (n times) $(a \in \mathbb{R}, n \in \mathbb{N})$
2. $a^0 = 1$ ($a \neq 0$ because 0^0 is undefined)
3. $a^{-n} = \frac{1}{a^n}$ ($a \neq 0$ because $\frac{1}{0}$ is undefined)

Examples:

1. $3 \times 3 = 3^2$
2. $5 \times 5 \times 5 \times 5 = 5^4$
3. $p \times p \times p = p^3$
4. $(3^x)^0 = 1$
5. $2^{-4} = \dfrac{1}{2^4} = \dfrac{1}{16}$
6. $\dfrac{1}{5^{-x}} = 5^x$

Notice that we always write the final answer with positive exponents.

Video: VMald at www.everythingmaths.co.za

3.1 Laws of exponents

EMAAG

There are several laws we can use to make working with exponential numbers easier. Some of these laws might have been done in earlier grades, but we list all the laws here for easy reference:

- $a^m \times a^n = a^{m+n}$
- $\frac{a^m}{a^n} = a^{m-n}$
- $(ab)^n = a^n b^n$
- $\left(\frac{a}{b}\right)^n = \frac{a^n}{b^n}$
- $(a^m)^n = a^{mn}$

where $a > 0$, $b > 0$ and $m, n \in \mathbb{Z}$.

Example 1: *Applying the exponential laws*

QUESTION

Simplify:

1. $2^{3x} \times 2^{4x}$
2. $\dfrac{12p^2t^5}{3pt^3}$
3. $(3x)^2$
4. $(3^4 5^2)^3$

SOLUTION

1. $2^{3x} \times 2^{4x} = 2^{3x+4x} = 2^{7x}$
2. $\frac{12p^2t^5}{3pt^3} = 4p^{(2-1)}t^{(5-3)} = 4pt^2$
3. $(3x)^2 = 3^2x^2 = 9x^2$
4. $(3^4 \times 5^2)^3 = 3^{(4\times 3)} \times 5^{(2\times 3)} = 3^{12} \times 5^6$

Example 2: *Exponential expressions*

QUESTION

Simplify: $\frac{2^{2n} \times 4^n \times 2}{16^n}$

SOLUTION

Step 1 : ***Change the bases to prime numbers***

$$\frac{2^{2n} \times 4^n \times 2}{16^n} = \frac{2^{2n} \times (2^2)^n \times 2^1}{(2^4)^n}$$

Step 2 : ***Simplify the exponents***

$$\begin{aligned} &= \frac{2^{2n} \times 2^{2n} \times 2^1}{2^{4n}} \\ &= \frac{2^{2n+2n+1}}{2^{4n}} \\ &= \frac{2^{4n+1}}{2^{4n}} \\ &= 2^{4n+1-(4n)} \\ &= 2 \end{aligned}$$

Example 3: *Exponential expressions*

QUESTION

Simplify: $\dfrac{5^{2x-1}9^{x-2}}{15^{2x-3}}$

SOLUTION

Step 1 : ***Change the bases to prime numbers***

$$\begin{aligned}\frac{5^{2x-1}9^{x-2}}{15^{2x-3}} &= \frac{5^{2x-1}(3^2)^{x-2}}{(5\times 3)^{2x-3}}\\ &= \frac{5^{2x-1}3^{2x-4}}{5^{2x-3}3^{2x-3}}\end{aligned}$$

Step 2 : ***Subtract the exponents (same base)***

$$\begin{aligned} &= 5^{(2x-1)-(2x-3)}\times 3^{(2x-4)-(2x-3)}\\ &= 5^{2x-1-2x+3}\times 3^{2x-4-2x+3}\\ &= 5^2\times 3^{-1}\end{aligned}$$

Step 3 : ***Write the answer as a fraction***

$$\begin{aligned} &= \frac{25}{3}\\ &= 8\frac{1}{3}\end{aligned}$$

Important: when working with exponents, all the laws of operation for algebra apply.

Example 4: *Simplifying by taking out a common factor*

QUESTION

Simplify: $\frac{2^t - 2^{t-2}}{3 \times 2^t - 2^t}$

SOLUTION

Step 1 : ***Simplify to a form that can be factorised***

$$\frac{2^t - 2^{t-2}}{3 \times 2^t - 2^t} = \frac{2^t - (2^t \times 2^{-2})}{3 \times 2^t - 2^t}$$

Step 2 : ***Take out a common factor***

$$= \frac{2^t(1 - 2^{-2})}{2^t(3 - 1)}$$

Step 3 : ***Cancel the common factor and simplify***

$$\begin{aligned} &= \frac{1 - \frac{1}{4}}{2} \\ &= \frac{\frac{3}{4}}{2} \\ &= \frac{3}{8} \end{aligned}$$

Example 5: *Simplifying using difference of two squares*

QUESTION

Simplify: $\frac{9^x - 1}{3^x + 1}$

SOLUTION

Step 1 : ***Change the bases to prime numbers***

$$\begin{aligned}\frac{9^x - 1}{3^x + 1} &= \frac{(3^2)^x - 1}{3^x + 1}\\ &= \frac{(3^x)^2 - 1}{3^x + 1}\end{aligned}$$

Step 2 : ***Factorise using the difference of squares***

$$= \frac{(3^x - 1)(3^x + 1)}{3^x + 1}$$

Step 3 : ***Simplify***

$$= 3^x - 1$$

Exercise 3 - 1

Simplify without using a calculator:

1. 16^0
2. $16a^0$
3. $\frac{2^{-2}}{3^2}$
4. $\frac{5}{2^{-3}}$
5. $\left(\frac{2}{3}\right)^{-3}$
6. x^2x^{3t+1}
7. $3 \times 3^{2a} \times 3^2$
8. $\frac{a^{3x}}{a^x}$
9. $\frac{32p^2}{4p^8}$
10. $(2t^4)^3$
11. $(3^{n+3})^2$
12. $\frac{3^n 9^{n-3}}{27^{n-1}}$

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(1.-12.) 00f0

3.2 Rational exponents EMAAH

We can also apply the exponential laws to expressions with rational exponents.

Video: VMaln at www.everythingmaths.co.za

Example 6: *Simplifying rational exponents*

QUESTION

Simplify: $2x^{\frac{1}{2}} \times 4x^{-\frac{1}{2}}$

SOLUTION

$$\begin{aligned} 2x^{\frac{1}{2}} \times 4x^{-\frac{1}{2}} &= 8x^{\frac{1}{2}-\frac{1}{2}} \\ &= 8x^0 \\ &= 8(1) \\ &= 8 \end{aligned}$$

Example 7: *Simplifying rational exponents*

QUESTION

Simplify: $(0{,}008)^{\frac{1}{3}}$

SOLUTION

Step 1 : ***Write as a fraction and change the bases to prime numbers***

$$\begin{aligned} 0{,}008 &= \frac{8}{1\ 000} \\ &= \frac{2^3}{10^3} \\ &= \left(\frac{2}{10}\right)^3 \end{aligned}$$

Step 2 : ***Therefore***

$$\begin{aligned} (0{,}008)^{\frac{1}{3}} &= \left[\left(\frac{2}{10}\right)^3\right]^{\frac{1}{3}} \\ &= \frac{2}{10} \\ &= \frac{1}{5} \end{aligned}$$

Exercise 3 - 2

Simplify without using a calculator:

1. $t^{\frac{1}{4}} \times 3t^{\frac{7}{4}}$
2. $\dfrac{16x^2}{(4x^2)^{\frac{1}{2}}}$
3. $(0{,}25)^{\frac{1}{2}}$
4. $(27)^{-\frac{1}{3}}$
5. $(3p^2)^{\frac{1}{2}} \times (3p^4)^{\frac{1}{2}}$

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(1.-5.) 00f1

3.3 Exponential equations

EMAAI

Exponential equations have the unknown variable in the exponent. Here are some examples:

$$\begin{aligned} 3^{x+1} &= 9 \\ 5^t + 3 \times 5^{t-1} &= 400 \end{aligned}$$

Solving exponential equations is simple: we need to apply the laws of exponents. This means that if we can write a single term with the same base on each side of the equation, we can equate the exponents.

Important: if $a > 0$ and $a \neq 1$

$$a^x = a^y$$
$$\text{then } x = y \quad \text{(same base)}$$

Also notice that if $a = 1$, then x and y can be different.

Example 8: *Equating exponents*

QUESTION

Solve for x: $3^{x+1} = 9$.

SOLUTION

Step 1 : ***Change the bases to prime numbers***

$$3^{x+1} = 3^2$$

Step 2 : ***The bases are the same so we can equate exponents***

$$\begin{aligned} x+1 &= 2 \\ \therefore x &= 1 \end{aligned}$$

Note: to solve exponential equations, we use all the strategies for solving linear and quadratic equations.

Example 9: *Solving equations by taking out a common factor*

QUESTION

Solve for t: $5^t + 3 \times 5^{t+1} = 400$.

SOLUTION

Step 1 : ***Rewrite the expression***

$$5^t + 3(5^t \times 5) = 400$$

Step 2 : ***Take out a common factor***

$$5^t(1+15) = 400$$

Step 3 : ***Simplify***

$$\begin{aligned} 5^t(16) &= 400 \\ 5^t &= 25 \end{aligned}$$

Step 4 : ***Change the bases to prime numbers***

$$5^t = 5^2$$

Step 5 : ***The bases are the same so we can equate exponents***

$$\therefore t = 2$$

Example 10: *Solving equations by factorising a trinomial*

QUESTION

Solve: $p - 13p^{\frac{1}{2}} + 36 = 0$.

SOLUTION

Step 1 : ***We notice that*** $(p^{\frac{1}{2}})^2 = p$ ***so we can rewrite the equation as***

$$(p^{\frac{1}{2}})^2 - 13p^{\frac{1}{2}} + 36 = 0$$

Step 2 : ***Factorise as a trinomial***

$$(p^{\frac{1}{2}} - 9)(p^{\frac{1}{2}} - 4) = 0$$

Step 3 : ***Solve to find both roots***

$$\begin{aligned} p^{\frac{1}{2}} - 9 &= 0 \\ p^{\frac{1}{2}} &= 9 \\ (p^{\frac{1}{2}})^2 &= (9)^2 \\ p &= 81 \end{aligned} \qquad \begin{aligned} p^{\frac{1}{2}} - 4 &= 0 \\ p^{\frac{1}{2}} &= 4 \\ (p^{\frac{1}{2}})^2 &= (4)^2 \\ p &= 16 \end{aligned}$$

Therefore $p = 81$ or $p = 16$.

Exercise 3 - 3

1. Solve for the variable:

   (a) $2^{x+5} = 32$
   (b) $5^{2x+2} = \frac{1}{125}$
   (c) $64^{y+1} = 16^{2y+5}$
   (d) $3^{9x-2} = 27$
   (e) $81^{k+2} = 27^{k+4}$
   (f) $25^{(1-2x)} - 5^4 = 0$
   (g) $27^x \times 9^{x-2} = 1$
   (h) $2^t + 2^{t+2} = 40$
   (i) $2 \times 5^{2-x} = 5 + 5^x$
   (j) $9^m + 3^{3-2m} = 28$
   (k) $y - 2y^{\frac{1}{2}} + 1 = 0$
   (l) $4^{x+3} = 0{,}5$
   (m) $2^a = 0{,}125$
   (n) $10^x = 0{,}001$
   (o) $2^{x^2-2x-3} = 1$

2. The growth of algae can be modelled by the function $f(t) = 2^t$. Find the value of t such that $f(t) = 128$.

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(1a-m.) 00f2 (2.) 0234

Chapter 3 | Summary

Summary presentation: VMdgh at www.everythingmaths.co.za

- Exponential notation means writing a number as a^n where n is an integer and a can be any real number.
- a is the base and n is the exponent or index.
- Definition:
  - $a^n = a \times a \times \cdots \times a$ (n times)
  - $a^0 = 1$, if $a \neq 0$
  - $a^{-n} = \frac{1}{a^n}$, if $a \neq 0$

- The laws of exponents:
  - $a^m \times a^n = a^{m+n}$
  - $\frac{a^m}{a^n} = a^{m-n}$
  - $(ab)^n = a^n b^n$
  - $\left(\frac{a}{b}\right)^n = \frac{a^n}{b^n}$
  - $(a^m)^n = a^{mn}$

Chapter 3 End of Chapter Exercises

1. Simplify:

(a) $t^3 \times 2t^0$

(b) $5^{2x+y}5^{3(x+z)}$

(c) $(b^{k+1})^k$

(d) $\frac{6^{5p}}{9^p}$

(e) $m^{-2t} \times (3m^t)^3$

(f) $\frac{3x^{-3}}{(3x)^2}$

(g) $\frac{5^{b-3}}{5^{b+1}}$

(h) $\frac{2^{a-2}3^{a+3}}{6^a}$

(i) $\frac{3^n 9^{n-3}}{27^{n-1}}$

(j) $\left(\frac{2x^{2a}}{y^{-b}}\right)^3$

(k) $\frac{2^{3x-1}8^{x+1}}{4^{2x-2}}$

(l) $\frac{6^{2x}11^{2x}}{22^{2x-1}3^{2x}}$

(m) $\frac{(-3)^{-3}(-3)^2}{(-3)^{-4}}$

(n) $(3^{-1} + 2^{-1})^{-1}$

(o) $\frac{9^{n-1}27^{3-2n}}{81^{2-n}}$

(p) $\frac{2^{3n+2}8^{n-3}}{4^{3n-2}}$

(q) $\frac{3^{t+3} + 3^t}{2 \times 3^t}$

(r) $\frac{2^{3p} + 1}{2^p + 1}$

2. Solve:

(a) $3^x = \frac{1}{27}$

(b) $5^{t-1} = 1$

(c) $2 \times 7^{3x} = 98$

(d) $2^{m+1} = (0{,}5)^{m-2}$

(e) $3^{y+1} = 5^{y+1}$

(f) $z^{\frac{3}{2}} = 64$

(g) $16x^{\frac{1}{2}} - 4 = 0$

(h) $m^0 + m^{-1} = 0$

(i) $t^{\frac{1}{2}} - 3t^{\frac{1}{4}} + 2 = 0$

(j) $3^p + 3^p + 3^p = 27$

(k) $k^{-1} - 7k^{-\frac{1}{2}} - 18 = 0$

(l) $x^{\frac{1}{2}} + 3x^{\frac{1}{4}} - 18 = 0$

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(1.) 00f3 (2.) 00f4

Number patterns 4

In earlier grades you saw patterns in the form of pictures and numbers. In this chapter, we learn more about the mathematics of patterns. Patterns are repetitive sequences and can be found in nature, shapes, events, sets of numbers and almost everywhere you care to look. For example, seeds in a sunflower, snowflakes, geometric designs on quilts or tiles, the number sequence $0; 4; 8; 12; 16; \ldots$.

See if you can spot any patterns in the following sequences:

1. $2;\ 4;\ 6;\ 8;\ 10; \ldots$
2. $1;\ 2;\ 4;\ 7;\ 11; \ldots$
3. $1;\ 4;\ 9;\ 16;\ 25; \ldots$
4. $5;\ 10;\ 20;\ 40;\ 80; \ldots$

Video: VMcyv at www.everythingmaths.co.za

Numbers can have interesting patterns. Here we examine some types of patterns and how they are formed.

Examples:

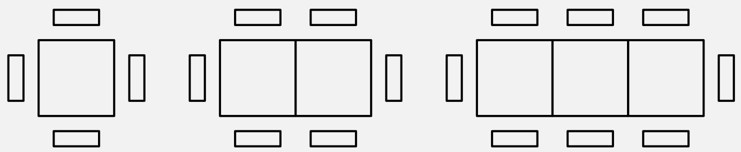
1. $1;\ 4;\ 7;\ 10;\ 13;\ 16;\ 19;\ 22;\ 25;\ \ldots$
   There is difference of 3 between successive terms.
   The pattern is continued by adding 3 to the last term.
2. $3;\ 8;\ 13;\ 18;\ 23;\ 28;\ 33;\ 38;\ \ldots$
   There is a difference of 5 between successive terms.
   The pattern is continued by adding 5 to the last term.
3. $2;\ 4;\ 8;\ 16;\ 32;\ 64;\ 128;\ 256;\ \ldots$
   This sequence has a factor of 2 between successive terms.
   The pattern is continued by multiplying the last term by 2.
4. $3;\ 9;\ 27;\ 81;\ 243;\ 729;\ 2187;\ \ldots$
   This sequence has a factor of 3 between successive terms.
   The pattern is continued by multiplying the last term by 3.

Example 1: *Study table*

QUESTION

You and 3 friends decide to study for Maths and are sitting together at a square table. A few minutes later, 2 other friends arrive and would like to sit at your table. You move another table next to yours so that 6 people can sit at the table. Another 2 friends also want to join your group, so you take a third table and add it to the existing tables. Now 8 people can sit together.

Examine how the number of people sitting is related to the number of tables. Is there a pattern?



Two more people can be seated for each table added.

SOLUTION

Step 1 : ***Make a table to see if a pattern forms***

Number of Tables, n	Number of people seated
1	$4 = 4$
2	$4 + 2 = 6$
3	$4 + 2 + 2 = 8$
4	$4 + 2 + 2 + 2 = 10$
$\vdots$	$\vdots$
n	$4 + 2 + 2 + 2 + \cdots + 2$

Step 2 : ***Describe the pattern***

We can see that for 3 tables we can seat 8 people, for 4 tables we can seat 10 people and so on. We started out with 4 people and added two each time. So for each table added, the number of people increased by 2.

4.1 Describing sequences

EMAAJ

To describe terms in a number pattern we use the following notation:

The 1^{st} term of a sequence is T_1.
The 4^{th} term of a sequence is T_4.
The 10^{th} term of a sequence is T_{10}.

The general term is often expressed as the n^{th} term and is written as T_n.

A sequence does not have to follow a pattern but, when it does, we can write down the general formula to calculate any term. For example, consider the following linear sequence:

$$1;\ 3;\ 5;\ 7;\ 9;\ \ldots$$

The n^{th} term is given by the general formula

$$T_n = 2n - 1$$

You can check this by substituting values into the formula:

$$\begin{aligned} T_1 &= 2(1) - 1 = 1 \\ T_2 &= 2(2) - 1 = 3 \\ T_3 &= 2(3) - 1 = 5 \\ T_4 &= 2(4) - 1 = 7 \\ T_5 &= 2(5) - 1 = 9 \end{aligned}$$

If we find the relationship between the position of a term and its value, we can describe the pattern and find any term in the sequence.

Video: MG10025 at www.everythingmaths.co.za

Common difference *EMAAK*

In some sequences, there is a constant difference between any two successive terms. This is called a *common difference*.

> ### *DEFINITION: Common difference*
>
> The common difference is the difference between any term and the term before it and is denoted by d.

For example, consider the sequence $10;\ 7;\ 4;\ 1;\ \ldots$
To calculate the common difference, we find the difference between any term and the previous term:

$$\begin{aligned} d &= T_2 - T_1 \\ &= 7 - 10 \\ &= -3 \end{aligned}$$

Let us check another two terms:

$$\begin{aligned} d &= T_4 - T_3 \\ &= 1 - 4 \\ &= -3 \end{aligned}$$

We see that d is constant.

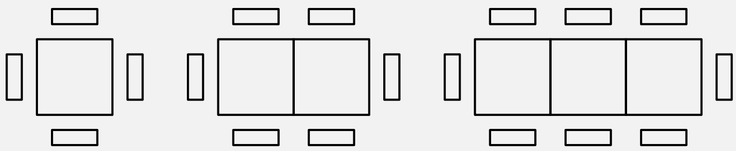
Important: $d = T_2 - T_1,$ not $T_1 - T_2$.

Example 2: *Study table, continued*

QUESTION

As before, you and 3 friends are studying for Maths and are sitting together at a square table. A few minutes later 2 other friends arrive so you move another table next to yours. Now 6 people can sit at the table. Another 2 friends also join your group, so you take a third table and add it to the existing tables. Now 8 people can sit together as shown below.

1. *Find the expression for the number of people seated at n tables.*
2. *Use the general formula to determine how many people can sit around 12 tables.*
3. *How many tables are needed to seat 20 people?*



Two more people can be seated for each table added.

SOLUTION

Step 1 : ***Make a table to see the pattern***

Number of Tables, n	**Number of people seated**	**Pattern**
1	$4 = 4$	$= 4 + 2(0)$
2	$4 + 2 = 6$	$= 4 + 2(1)$
3	$4 + 2 + 2 = 8$	$= 4 + 2(2)$
4	$4 + 2 + 2 + 2 = 10$	$= 4 + 2(3)$
$\vdots$	$\vdots$	$\vdots$
n	$4 + 2 + 2 + 2 + \cdots + 2$	$= 4 + 2(n - 1)$

Step 2 : ***Describe the pattern***

The number of people seated at n tables is $T_n = 4 + 2(n-1)$.

Step 3 : ***Calculate the*** 12^{th} ***term, in other words, find*** T_n ***if*** $n = 12$

$$\begin{aligned} T_{12} &= 4 + 2(12-1) \\ &= 4 + 2(11) \\ &= 4 + 22 \\ &= 26 \end{aligned}$$

Therefore 26 people can be seated at 12 tables.

Step 4 : ***Calculate the number of tables needed to seat*** 20 ***people, in other words, find*** n ***if*** $T_n = 20$

$$\begin{aligned} T_n &= 4 + 2(n-1) \\ 20 &= 4 + 2(n-1) \\ 20 - 4 &= 2(n-1) \\ \frac{16}{2} &= n - 1 \\ 8 + 1 &= n \\ n &= 9 \end{aligned}$$

Therefore 9 tables are needed to seat 20 people.

It is important to note the difference between n and T_n. n can be compared to a place holder indicating the position of the term in the sequence, while T_n is the value of the place held by n. From the example above, the first table holds 4 people. So for $n = 1$, the value of $T_1 = 4$ and so on:

n	1	2	3	4	...
T_n	4	6	8	10	...

Exercise 4 - 1

1. Write down the next three terms in each of the following sequences:
   (a) $5;\ 15;\ 25;\ \ldots$
   (b) $-8; -3;\ 2 : `\ldots$
   (c) $30;\ 27;\ 24;\ \ldots$
2. The general term is given for each sequence below. Calculate the missing terms.
   (a) $0; 3;\ \ldots;\ 15;\ 24$ $\qquad T_n = n^2 - 1$
   (b) $3;\ 2;\ 1;\ 0;\ \ldots;\ -2$ $\qquad T_n = -n + 4$
   (c) $-11;\ \ldots;\ -7;\ \ldots;\ -3$ $\qquad T_n = -13 + 2n$
3. Find the general formula for the following sequences and then find T_{10}, T_{50} and T_{100}
   (a) $2;\ 5;\ 8;\ 11;\ 14;\ \ldots$
   (b) $0;\ 4;\ 8;\ 12;\ 16;\ \ldots$
   (c) $2;\ -1;\ -4;\ -7;\ -10;\ \ldots$

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(1.) 00i0 (2.) 00i1 (3.) 00i2

4.2 Patterns and conjecture EMAAL

In mathematics, a conjecture is a mathematical statement which appears to be true, but has not been formally proven. A conjecture can be thought of as the mathematicians way of saying "I believe that this is true, but I have no proof yet". A conjecture is a good guess or an idea about a pattern.

For example, make a conjecture about the next number in the pattern $2;\ 6;\ 11;\ 17; \ldots$ The terms increase by 4, then 5, and then 6. Conjecture: the next term will increase by 7, so it will be $17 + 7 = 24$.

Video: MG10026 at www.everythingmaths.co.za

Example 3: *Adding even and odd numbers*

QUESTION

1. *Investigate the type of number you get if you find the sum of an odd number and an even number.*
2. *Express your answer in words as a conjecture.*
3. *Use algebra to prove this conjecture.*

SOLUTION

Step 1 : ***First try some examples***

$$\begin{aligned} 23 + 12 &= 35 \\ 148 + 31 &= 179 \\ 11 + 200 &= 211 \end{aligned}$$

Step 2 : ***Make a conjecture***

The sum of any odd number and any even number is always odd.

Step 3 : ***Express algebraically***

Express the even number as $2x$.

Express the odd number as $2y - 1$.

$$\begin{aligned} \text{Sum} &= 2x + (2y - 1) \\ &= 2x + 2y - 1 \\ &= (2x + 2y) - 1 \\ &= 2(x + y) - 1 \end{aligned}$$

From this we can see that $2(x + y)$ is an even number. So then $2(x + y) - 1$ is an odd number. Therefore our conjecture is true.

Example 4: *Multiplying a two-digit number by* 11

QUESTION

1. *Consider the following examples of multiplying any two-digit number by* 11. *What pattern can you see?*

$$\begin{aligned} 11 \times 42 &= 462 \\ 11 \times 71 &= 781 \\ 11 \times 45 &= 495 \end{aligned}$$

2. *Express your answer as a conjecture.*
3. *Can you find an example that disproves your initial conjecture? If so, reconsider your conjecture.*
4. *Use algebra to prove your conjecture.*

SOLUTION

Step 1 : ***Find the pattern***

We notice in the answer that the middle digit is the sum of the two digits in the original two-digit number.

Step 2 : ***Make a conjecture***

The middle digit of the product is the sum of the two digits of the original number that is multiplied by 11.

Step 3 : ***Reconsidering the conjecture***

We notice that

$$\begin{aligned} 11 \times 67 &= 737 \\ 11 \times 56 &= 616 \end{aligned}$$

Therefore our conjecture only holds true if the sum of the two digits is less than 10.

Step 4 : ***Express algebraically***

Any two-digit number can be written as $10a + b$. For example, $34 = 10(3) + 4$. Any three-digit number can be written as $100a + 10b + c$.

For example, $582 = 100(5) + 10(8) + 2$.

$$\begin{aligned}
11 \times (10x + y) &= 110x + 11y \\
&= (100x + 10x) + 10y + y \\
&= 100x + (10x + 10y) + y \\
&= 100x + 10(x + y) + y
\end{aligned}$$

From this equation we can see that the middle digit of the three-digit number is equal to the sum of the two digits x and y.

Chapter 4 | Summary

Summary presentation: VMcyz at www.everythingmaths.co.za

- There are several special sequences of numbers.
  - Triangular numbers: $1;\ 3;\ 6;\ 10;\ 15;\ 21;\ 28;\ 36;\ 45;\ \ldots$
  - Square numbers: $1;\ 4;\ 9;\ 16;\ 25;\ 36;\ 49;\ 64;\ 81;\ \ldots$
  - Cube numbers: $1;\ 8;\ 27;\ 64;\ 125;\ 216;\ 343;\ 512;\ 729;\ \ldots$
- The general term is expressed as the n^{th} term and written as T_n.
- We define the common difference d of a sequence as the difference between any two successive terms.
- We can work out a general formula for each number pattern and use it to determine any term in the pattern.
- A conjecture is something that you believe to be true but have not yet proven.

Chapter 4 End of Chapter Exercises

1. Find the 6^{th} term in each of the following sequences:
   (a) $4;\ 13;\ 22;\ 31;\ \ldots$
   (b) $5;\ 2;\ -1;\ -4;\ \ldots$

   (c) $7{,}4;\ 9{,}7;\ 12;\ 14{,}3;\ \ldots$

2. Find the general term of the following sequences:

   (a) $3;\ 7;\ 11;\ 15;\ \ldots$

   (b) $-2;\ 1;\ 4;\ 7;\ \ldots$

   (c) $11;\ 15;\ 19;\ 23;\ \ldots$

   (d) $\frac{1}{3};\ \frac{2}{3};\ 1;\ 1\frac{1}{3};\ \ldots$

3. The seating of a sports stadium is arranged so that the first row has 15 seats, the second row has 19 seats, the third row has 23 seats and so on. Calculate how many seats are in the twenty-fifth row.

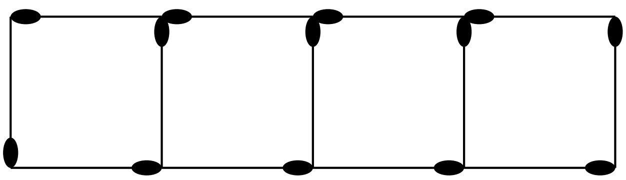
4. A single square is made from 4 matchsticks. Two squares in a row need 7 matchsticks and three squares in a row need 10 matchsticks. For this sequence determine:

   (a) the first term;

   (b) the common difference;

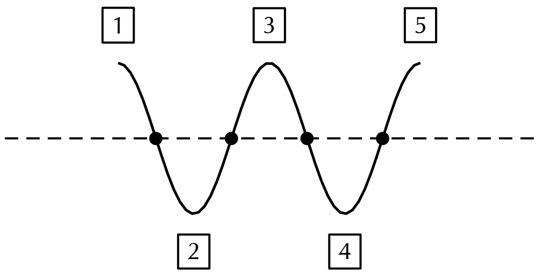
   (c) the general formula;

   (d) how many matchsticks there are in a row of twenty-five squares.



5. You would like to start saving some money, but because you have never tried to save money before, you decide to start slowly. At the end of the first week you deposit R 5 into your bank account. Then at the end of the second week you deposit R 10 and at the end of the third week, R 15. After how many weeks will you deposit R 50 into your bank account?

6. A horizontal line intersects a piece of string at 4 points and divides it into five parts, as shown below.



   If the piece of string is intersected in this way by 19 parallel lines, each of which intersects it at 4 points, determine the number of parts into which the string will be divided.

7. Consider what happens when you add 9 to a two-digit number:

$$
\begin{aligned}
9 + 16 &= 25 \\
9 + 28 &= 37 \\
9 + 43 &= 52
\end{aligned}
$$

(a) What pattern do you see?

(b) Make a conjecture and express it in words.

(c) Generalise your conjecture algebraically.

(d) Prove that your conjecture is true.

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(1a-c.) 00i3 (2a.) 00i4 (2b.) 00i5 (2c.) 00i6 (2d.) 00i7

(3.) 00i8 (4a-d.) 00i9 (5.) 00ia (6.) 00ib (7.) 00ic

Functions 5

5.1 Functions in the real world

EMAAM

Functions are mathematical building blocks for designing machines, predicting natural disasters, curing diseases, understanding world economies and for keeping aeroplanes in the air. Functions can take input from many variables, but always give the same output, unique to that function.

Functions also allow us to visualise relationships in terms of graphs, which are much easier to read and interpret than lists of numbers.

Some examples of functions include:

- Money as a function of time. You never have more than one amount of money at any time because you can always add everything to give one total amount. By understanding how your money changes over time, you can plan to spend your money sensibly. Businesses find it very useful to plot the graph of their money over time so that they can see when they are spending too much.
- Temperature as a function of various factors. Temperature is a very complicated function because it has so many inputs, including: the time of day, the season, the amount of clouds in the sky, the strength of the wind, where you are and many more. But the important thing is that there is only one temperature output when you measure it in a specific place.
- Location as a function of time. You can never be in two places at the same time. If you were to plot the graphs of where two people are as a function of time, the place where the lines cross means that the two people meet each other at that time. This idea is used in logistics, an area of mathematics that tries to plan where people and items are for businesses.

Video: VMawb at www.everythingmaths.co.za

DEFINITION: Function

A function is a mathematical relationship between two variables, where every input variable has one output variable.

Dependent and independent variables

EMAAN

In functions, the x-variable is known as the input or independent variable, because its value can be chosen freely. The calculated y-variable is known as the output or dependent variable, because its value depends on the chosen input value.

Set notation

EMAAO

Examples:

$\{x : x \in \mathbb{R}, x > 0\}$	The set of all x-values such that x is an element of the set of real numbers and is greater than 0.
$\{y : y \in \mathbb{N}, 3 < y \leq 5\}$	The set of all y-values such that y is a natural number, is greater than 3 and is less than or equal to 5.
$\{z : z \in \mathbb{Z}, z \leq 100\}$	The set of all z-values such that z is an integer and is less than or equal to 100.

Interval notation

EMAAP

It is important to note that this notation can only be used to represent an interval of real numbers.

Examples:

$(3;11)$	Round brackets indicate that the number is not included. This interval includes all real numbers greater than but not equal to 3 and less than but not equal to 11.
$(-\infty;-2)$	Round brackets are always used for positive and negative infinity. This interval includes all real numbers less than, but not equal to -2.
$[1;9)$	A square bracket indicates that the number is included. This interval includes all real numbers greater than or equal to 1 and less than but not equal to 9.

Function notation

EMAAQ

This is a very useful way to express a function. Another way of writing $y = 2x + 1$ is $f(x) = 2x + 1$. We say "f of x is equal to $2x + 1$". Any letter can be used, for example, $g(x)$, $h(x)$, $p(x)$, etc.

1. **Determine the output value**:
   "Find the value of the function for $x = -3$" can be written as: "find $f(-3)$".
   Replace x with -3:

$$\begin{aligned} f(-3) &= 2(-3) + 1 = -5 \\ \therefore f(-3) &= -5 \end{aligned}$$

This means that when $x = -3$, the value of the function is -5.

2. **Determine the input value**:

   "Find the value of x that will give a y-value of 27" can be written as: "find x if $f(x) = 27$".

   We write the following equation and solve for x:

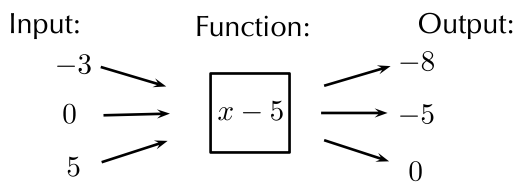
$$\begin{aligned} 2x + 1 &= 27 \\ \therefore x &= 13 \end{aligned}$$

   This means that when $x = 13$ the value of the function is 27.

Representations of functions *EMAAR*

Functions can be expressed in many different ways for different purposes.

1. Words: "The relationship between two variables is such that one is always 5 less than the other."
2. Mapping diagram:

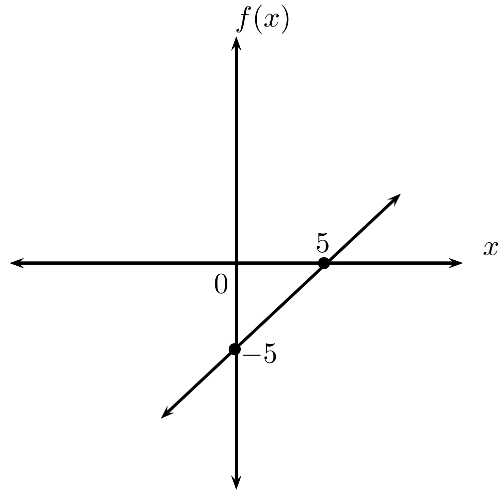


3. Table:

Input variable (x)	-3	0	5
Output variable (y)	-8	-5	0

4. Set of ordered number pairs: $(-3; -8)$, $(0; -5)$, $(5; 0)$
5. Algebraic formula: $f(x) = x - 5$

6. Graph:



Domain and range

EMAAS

The domain of a function is the set of all independent x-values for which there is one dependent y-value according to that function. The range is the set of all dependent y-values which can be obtained using an independent x-value.

Exercise 5 - 1

1. Write the following in set notation:
   - **(a)** $(-\infty; 7]$
   - **(b)** $[13; 4)$
   - **(c)** $(35; \infty)$
   - **(d)** $[\frac{3}{4}; 21)$
   - **(e)** $[-\frac{1}{2}; \frac{1}{2}]$
   - **(f)** $(-\sqrt{3}; \infty)$
2. Write the following in interval notation:
   - **(a)** $\{p : p \in \mathbb{R}, p \leq 6\}$
   - **(b)** $\{k : k \in \mathbb{R}, \ -5 < k < 5\}$
   - **(c)** $\{x : x \in \mathbb{R}, x > \frac{1}{5}\}$
   - **(d)** $\{z : z \in \mathbb{R}, 21 \leq z < 41\}$

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(1a-f.) 00f9 (2a-d.) 00fa

5.2 Linear functions EMAAT

Functions of the form $y = x$ EMAAU

Functions of the form $y = mx + c$ are called straight line functions. In the equation, $y = mx + c$, m and c are constants and have different effects on the graph of the function.
Video: VMjlm at www.everythingmaths.co.za

Example 1: *Plotting a straight line graph*

QUESTION

$$y = f(x) = x$$

Complete the following table for $f(x) = x$ and plot the points on a set of axes.

x	-2	-1	0	1	2
$f(x)$	-2				

1. *Join the points with a straight line.*
2. *Determine the domain and range.*
3. *About which line is f symmetrical?*
4. *Using the graph, determine the value of x for which $f(x) = 4$. Confirm your answer graphically.*
5. *Where does the graph cut the axes?*

SOLUTION

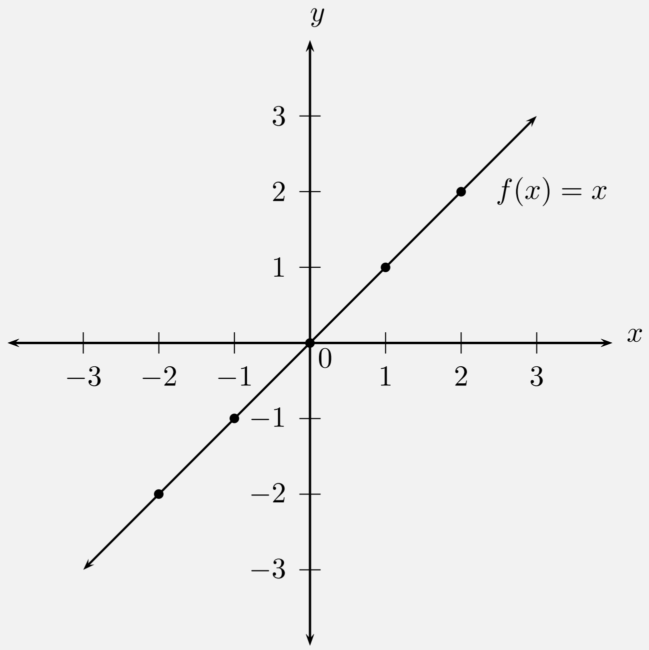
Step 1 : ***Substitute values into the equation***

x	-2	-1	0	1	2
$f(x)$	-2	-1	0	1	2

Step 2 : ***Plot the points and join with a straight line curve***

From the table, we get the following points and the graph:

$$(-2;-2),\ (-1;-1),\ (0;0),\ (1;1),\ (2;2)$$



Step 3 : ***Determine the domain and range***

Domain: $x \in \mathbb{R}$

Range: $f(x) \in \mathbb{R}$

Step 4 : ***Determine the value of x for which $f(x) = 4$***

From the graph we see that when $f(x) = 4$, $x = 4$. This gives the point $(4; 4)$.

Step 5 : ***Determine the intercept***

The function f intercepts the axes at the origin $(0;0)$.

Functions of the form $y = mx + c$

EMAAV

Investigation:	*The effects of m and c on a straight line graph*

On the same set of axes, plot the following graphs:

1. $y = x - 2$
2. $y = x - 1$
3. $y = x$
4. $y = x + 1$
5. $y = x + 2$

Use your results to deduce the effect of different values of c on the graph. On the same set of axes, plot the following graphs:

6. $y = -2x$
7. $y = -x$
8. $y = x$
9. $y = 2x$

Use your results to deduce the effect of different values of m on the graph.

The effect of m

We notice that the value of m affects the slope of the graph. As m increases, the gradient of the graph increases.

If $m > 0$ then the graph increases from left to right (slopes upwards).

If $m < 0$ then the graph increases from right to left (slopes downwards). For this reason, m is referred to as the gradient of a straight-line graph.

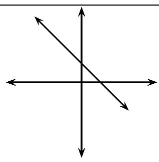
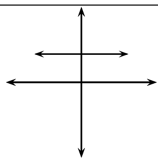
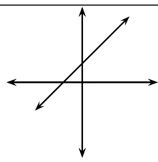
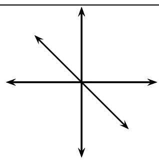
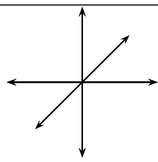
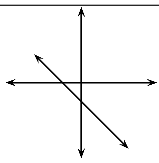
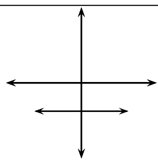
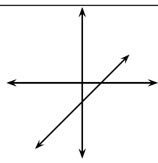
The effect of c

We also notice that the value of c affects where the graph cuts the y-axis. For this reason, c is known as the y-intercept.

If $c > 0$ the graph shifts vertically upwards.

If $c < 0$, the graph shifts vertically downwards.

Video: VMaxf at www.everythingmaths.co.za

	$m < 0$	$m = 0$	$m > 0$
$c > 0$			
$c = 0$			
$c < 0$			

Discovering the characteristics

EMAAW

The standard form of a straight line graph is the equation $y = mx + c$.

Domain and range

The domain is $\{x : x \in \mathbb{R}\}$ because there is no value of x for which $f(x)$ is undefined. The range of $f(x) = mx + c$ is also $\{f(x) : f(x) \in \mathbb{R}\}$ because $f(x)$ can take on any real value.

Intercepts

The y-intercept:
Every point on the y-axis has an x-coordinate of 0. Therefore to calculate the y-intercept, let $x = 0$.

For example, the y-intercept of $g(x) = x - 1$ is given by setting $x = 0$:

$$\begin{aligned} g(x) &= x-1 \\ g(0) &= 0-1 \\ &= -1 \end{aligned}$$

This gives the point $(0;-1)$.

The x-intercept:
Every point on the x-axis has a y-coordinate of 0. Therefore to calculate the x-intercept, let $y=0$. For example, the x-intercept of $g(x)=x-1$ is given by setting $y=0$:

$$\begin{aligned} g(x) &= x-1 \\ 0 &= x-1 \\ \therefore x &= 1 \end{aligned}$$

This gives the point $(1;0)$.

Sketching graphs of the form $y = mx + c$ EMAAX

In order to sketch graphs of the form, $f(x) = mx + c$, we need to determine three characteristics:

1. sign of m
2. y-intercept
3. x-intercept

Dual intercept method EMAAY

Only two points are needed to plot a straight line graph. The easiest points to use are the x-intercept and the y-intercept.

Example 2: *Sketching a straight line graph using the dual intercept method*

QUESTION

Sketch the graph of $g(x) = x - 1$ using the dual intercept method.

SOLUTION

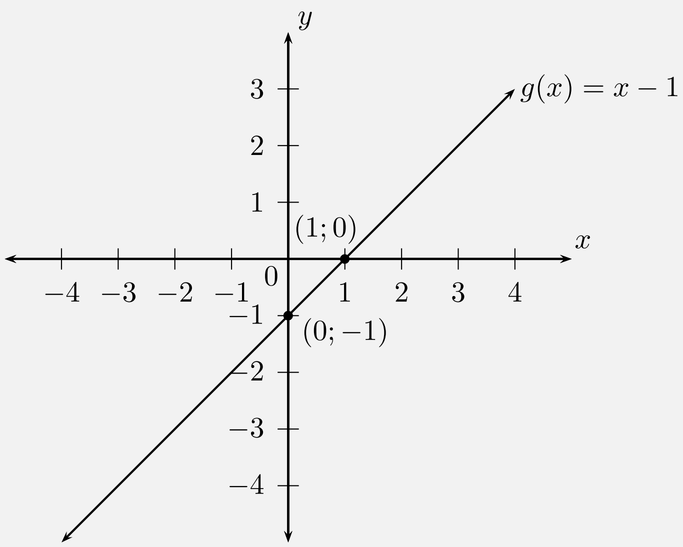
Step 1 : ***Examine the standard form of the equation***

$m > 0$. This means that the graph increases as x increases.

Step 2 : ***Calculate the intercepts***

For y-intercept, let $x = 0$; therefore $g(0) = -1$. This gives the point $(0; -1)$. For x-intercept, let $y = 0$; therefore $x = 1$. This gives the point $(1; 0)$.

Step 3 : ***Plot the points and draw the graph***



Gradient and y-intercept method EMAAZ

We can draw a straight line graph of the form $y = mx + c$ using the gradient m and the y-intercept c.

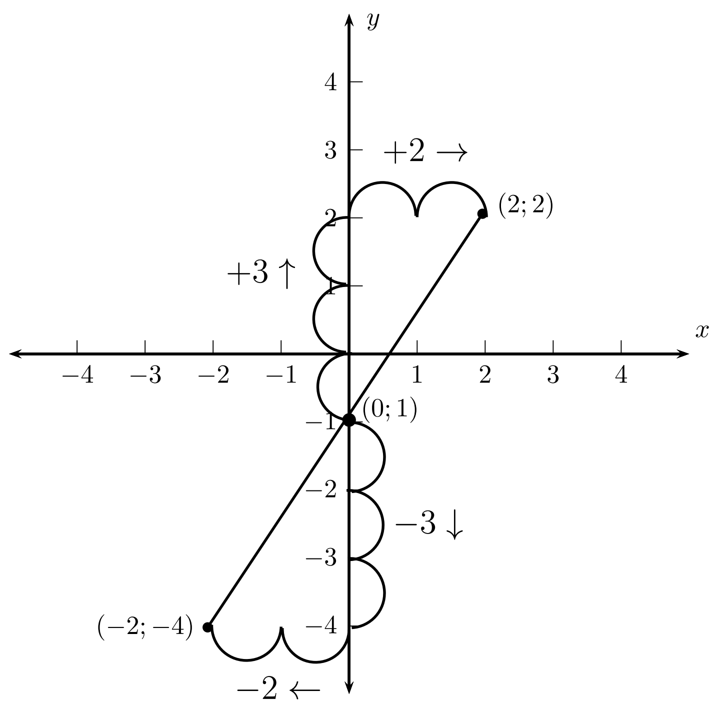
We calculate the y-intercept by letting $x = 0$. This gives us one point $(0; c)$ for drawing the graph and we use the gradient (m) to calculate the second.

The gradient of a line is the measure of steepness. Steepness is determined by the ratio of vertical change to horizontal change:

$$m = \frac{\text{change in } y}{\text{change in } x} = \frac{\text{vertical change}}{\text{horizontal change}}$$

For example, $y = \frac{3}{2}x - 1$, therefore $m > 0$ and the graph slopes upwards.

$$m = \frac{\text{change in } y}{\text{change in } x} = \frac{3\uparrow}{2\rightarrow} = \frac{-3\downarrow}{-2\leftarrow}$$



Example 3: *Sketching a straight line graph using the gradient–intercept method*

QUESTION

Sketch the graph of $p(x) = \frac{1}{2}x - 3$ using the gradient–intercept method.

SOLUTION

Step 1 : ***Use the intercept***

$c = -3$, which gives the point $(0; -3)$.

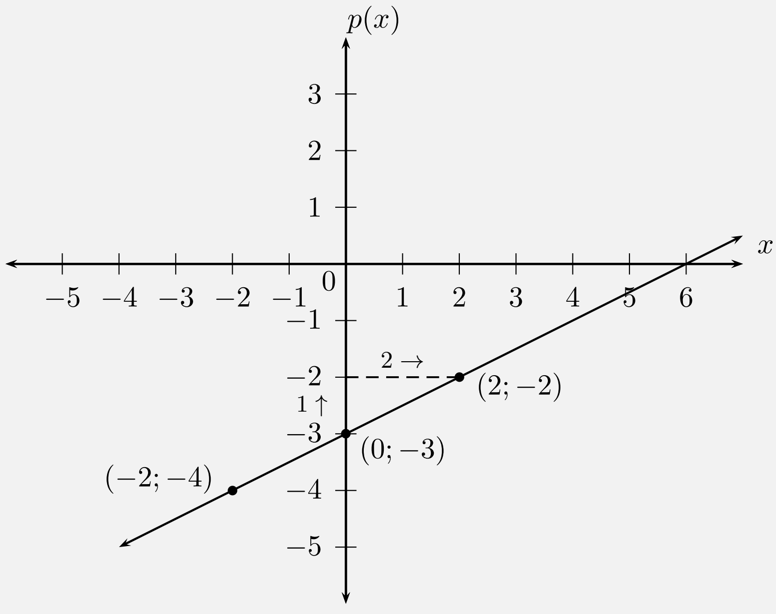
Step 2 : ***Use the gradient***

$$m = \frac{\text{change in } y}{\text{change in } x} = \frac{1 \uparrow}{2 \rightarrow} = \frac{-1 \downarrow}{-2 \leftarrow}$$

Start at $(0; -3)$. Move 1 unit up and 2 units to the right. This gives the second point $(2; -2)$.

Or start at $(0; -3)$, move 1 unit down and 2 units to the left. This gives the second point $(-2; -4)$.

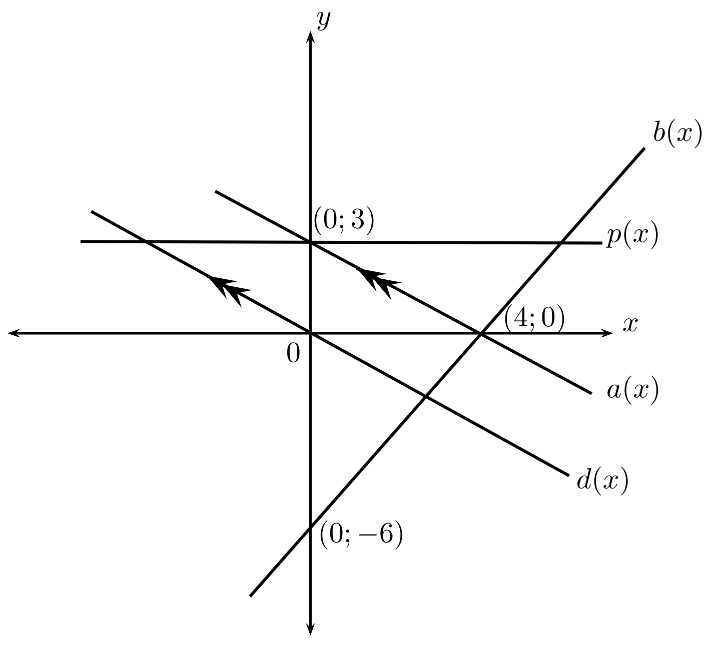
Step 3 : ***Plot the points and draw the graph***



Always write the function in the form $y = mx + c$ and take note of m. After plotting the graph, make sure that the graph increases if $m > 0$ and that the graph decreases if $m < 0$.

Exercise 5 - 2

1. List the x and y-intercepts for the following straight line graphs. Indicate whether the graph is increasing or decreasing:
   - **(a)** $y = x + 1$
   - **(b)** $y = x - 1$
   - **(c)** $h(x) = 2x - 1$
   - **(d)** $y + 3x = 1$
   - **(e)** $3y - 2x = 6$
   - **(f)** $k(x) = -3$
   - **(g)** $x = 3y$
   - **(h)** $\frac{x}{2} - \frac{y}{3} = 1$
2. For the functions in the diagram below, give the equation of the line:
   - **(a)** $a(x)$
   - **(b)** $b(x)$
   - **(c)** $p(x)$
   - **(d)** $d(x)$



3. Sketch the following functions on the same set of axes, using the dual intercept method. Clearly indicate the intercepts and the point of intersection of the two graphs: $x + 2y - 5 = 0$ and $3x - y - 1 = 0$
4. On the same set of axes, draw the graphs of $f(x) = 3 - 3x$ and $g(x) = \frac{1}{3}x + 1$ using the gradient–intercept method.

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(1a-h.) 00fh (2a-d.) 00fi (3.) 00fj (4.) 00fk

5.3 Quadratic functions EMABA

Functions of the form $y = x^2$ EMABB

Functions of the general form $y = ax^2 + q$ are called parabolic functions. In the equation $y = ax^2 + q$, a and q are constants and have different effects on the parabola.

Video: VMaxl at www.everythingmaths.co.za

Example 4: *Plotting a quadratic function*

QUESTION

$$y = f(x) = x^2$$

Complete the following table for $f(x) = x^2$ and plot the points on a system of axes.

x	-3	-2	-1	0	1	2	3
$f(x)$	9						

1. *Join the points with a smooth curve.*
2. *The domain of f is $x \in \mathbb{R}$. Determine the range.*
3. *About which line is f symmetrical?*
4. *Determine the value of x for which $f(x) = 6\frac{1}{4}$.*
   Confirm your answer graphically.
5. *where does the graph cut the axes?*

SOLUTION

Step 1 : ***Substitute values into the equation***

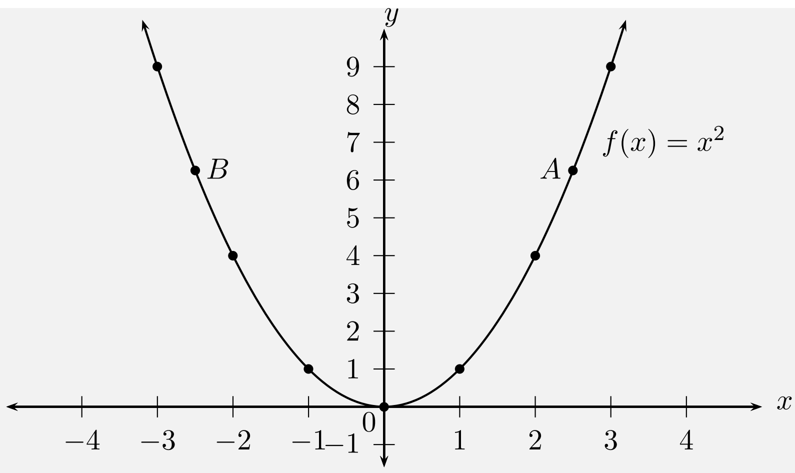
$$
\begin{array}{rcccc}
f(x) & = & x^2 & & \\
f(-3) & = & (-3)^2 & = & 9 \\
f(-2) & = & (-2)^2 & = & 4 \\
f(-1) & = & (-1)^2 & = & 1 \\
f(0) & = & (0)^2 & = & 0 \\
f(1) & = & (1)^2 & = & 1 \\
f(2) & = & (2)^2 & = & 4 \\
f(3) & = & (3)^2 & = & 9
\end{array}
$$

x	-3	-2	-1	0	1	2	3
$f(x)$	9	4	1	0	1	4	9

Step 2 : ***Plot the points and join with a smooth curve***

From the table, we get the following points:

$$(-3;9),\ (-2;4),\ (-1;1),\ (0;0),\ (1;1),\ (2;4),\ (3;9)$$



Step 3 : ***Determine the domain and range***

Domain: $x \in \mathbb{R}$.

From the graph we see that for all values of x, $y \geq 0$.

Range: $\{y : y \in \mathbb{R},\ y \geq 0\}$.

Step 4 : ***Find the axis of symmetry***

f is symmetrical about the y-axis. Therefore the axis of symmetry of f is the line $x = 0$.

Step 5 : ***Determine the x-value***

$$\begin{aligned} f(x) &= \tfrac{25}{4} \\ \therefore \tfrac{25}{4} &= x^2 \\ x &= \pm\tfrac{5}{2} \\ &= \pm 2\tfrac{1}{2} \end{aligned}$$

See points A and B on the graph.

Step 6 : ***Determine the intercept***

The function f intercepts the axes at the origin $(0; 0)$.

We notice that as the value of x increases from $-\infty$ to 0, $f(x)$ decreases.

At the turning point $(0; 0)$, $f(x) = 0$.

As the value of x increases from 0 to ∞, $f(x)$ increases.

Functions of the form $y = ax^2 + q$ EMABC

Investigation:	*The effects of a and q on a parabola*

Complete the table and plot the following graphs on the same system of axes:

1. $y_1 = x^2 - 2$
2. $y_2 = x^2 - 1$
3. $y_3 = x^2$
4. $y_4 = x^2 + 1$
5. $y_5 = x^2 + 2$

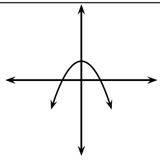
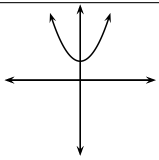
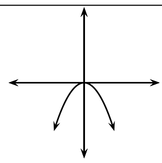
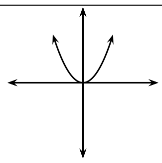
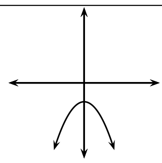
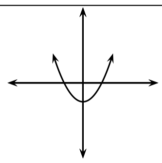
x	-2	-1	0	1	2
y_1					
y_2					
y_3					
y_4					
y_5					

Use your results to deduce the effect of q. Complete the table and plot the following graphs on the same system of axes:

6. $y_6 = -2x^2$
7. $y_7 = -x^2$
8. $y_8 = x^2$
9. $y_9 = 2x^2$

x	-2	-1	0	1	2
y_6					
y_7					
y_8					
y_9					

Use your results to deduce the effect of a.

	$a < 0$	$a > 0$
$q > 0$		
$q = 0$		
$q < 0$		

The effect of q

The effect of q is called a vertical shift because all points are moved the same distance in the same direction (it slides the entire graph up or down).

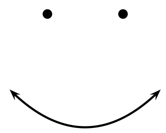
- For $q > 0$, the graph of $f(x)$ is shifted vertically upwards by q units. The turning point of $f(x)$ is above the y-axis.
- For $q < 0$, the graph of $f(x)$ is shifted vertically downwards by q units. The turning point of $f(x)$ is below the y-axis.

The effect of a

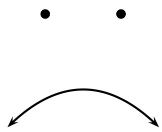
The sign of a determines the shape of the graph.

- For $a > 0$, the graph of $f(x)$ is a "smile" and has a minimum turning point at $(0; q)$. The graph of $f(x)$ is stretched vertically upwards; as a gets larger, the graph gets narrower.
  For $0 < a < 1$, as a gets closer to 0, the graph of $f(x)$ get wider.
- For $a < 0$, the graph of $f(x)$ is a "frown" and has a maximum turning point at $(0; q)$. The graph of $f(x)$ is stretched vertically downwards; as a gets smaller, the graph gets narrower.
  For $-1 < a < 0$, as a gets closer to 0, the graph of $f(x)$ get wider.

Simulation: MG10018 at www.everythingmaths.co.za



$a > 0$ (a positive smile)



$a < 0$ (a negative frown)

Discovering the characteristics

EMABD

The standard form of a parabola is the equation $y = ax^2 + q$.

Domain and range

The domain is $\{x : x \in \mathbb{R}\}$ because there is no value for which $f(x)$ is undefined.

If $a > 0$ then we have:

$$\begin{aligned} x^2 &\geq 0 && \text{(Perfect square is always positive)} \\ ax^2 &\geq 0 && \text{(since } a > 0\text{)} \\ ax^2 + q &\geq q && \text{(add } q \text{ to both sides)} \\ \therefore f(x) &\geq q \end{aligned}$$

Therefore if $a > 0$, the range is $[q; \infty)$. Similarly, if $a < 0$ then the range is $(-\infty; q]$.

Example 5: *Domain and range of a parabola*

QUESTION

If $g(x) = x^2 + 2$, determine the domain and range of the function.

SOLUTION

Step 1 : ***Determine the domain***

The domain is $\{x : x \in \mathbb{R}\}$ because there is no value for which $g(x)$ is undefined.

Step 2 : ***Determine the range***

The range of $g(x)$ can be calculated as follows:

$$\begin{aligned} x^2 &\geq 0 \\ x^2+2 &\geq 2 \\ g(x) &\geq 2 \end{aligned}$$

Therefore the range is $\{g(x) : g(x) \geq 2\}$.

Intercepts

The y-intercept:
Every point on the y-axis has an x-coordinate of 0, therefore to calculate the y-intercept let $x = 0$.
For example, the y-intercept of $g(x) = x^2 + 2$ is given by setting $x = 0$:

$$\begin{aligned} g(x) &= x^2+2 \\ g(0) &= 0^2+2 \\ &= 2 \end{aligned}$$

This gives the point $(0; 2)$.

The x-intercepts:
Every point on the x-axis has a y-coordinate of 0, therefore to calculate the x-intercepts let $y = 0$.
For example, the x-intercepts of $g(x) = x^2 + 2$ are given by setting $y = 0$:

$$\begin{aligned} g(x) &= x^2+2 \\ 0 &= x^2+2 \\ -2 &= x^2 \end{aligned}$$

There is no real solution, therefore the graph of $g(x) = x^2 + 2$ does not have any x-intercepts.

Turning points

The turning point of the function of the form $f(x) = ax^2 + q$ is determined by examining the range of the function.

- If $a > 0$, the graph of $f(x)$ is a "smile" and has a minimum turning point at $(0; q)$.
- If $a < 0$, the graph of $f(x)$ is a "frown" and has a maximum turning point at $(0; q)$.

Axes of symmetry

The axis of symmetry for functions of the form $f(x) = ax^2 + q$ is the y-axis, which is the line $x = 0$.

Sketching graphs of the form $y = ax^2 + q$ *EMABE*

In order to sketch graphs of the form $f(x) = ax^2 + q$, we need to determine the following characteristics:

1. sign of a
2. y-intercept
3. x-intercept
4. turning point

Example 6: *Sketching a parabola*

QUESTION

Sketch the graph of $y = 2x^2 - 4$. Mark the intercepts and the turning point.

SOLUTION

Step 1 : ***Examine the standard form of the equation***

We notice that $a > 0$. Therefore the graph of the function is a "smile" and has a minimum turning point.

Step 2 : ***Calculate the intercepts***

For the y-intercept, let $x = 0$:

$$\begin{aligned} y &= 2x^2 - 4 \\ &= 2(0)^2 - 4 \\ &= -4 \end{aligned}$$

This gives the point $(0; -4)$.
For x-intercept, let $y = 0$:

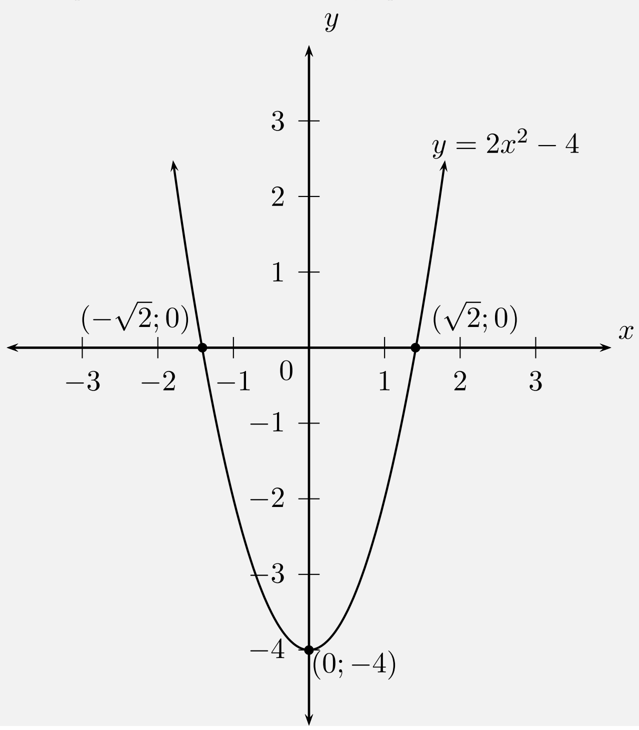
$$\begin{aligned} y &= 2x^2 - 4 \\ 0 &= 2x^2 - 4 \\ x^2 &= 2 \\ \therefore x &= \pm\sqrt{2} \end{aligned}$$

This gives the points $(-\sqrt{2}; 0)$ and $(\sqrt{2}; 0)$.

Step 3 : ***Determine the turning point***

From the standard form of the equation we see that the turning point is $(0; -4)$.

Step 4 : ***Plot the points and sketch the graph***



Domain: $\{x : x \in \mathbb{R}\}$
Range: $\{y : y \geq -4, y \in \mathbb{R}\}$
The axis of symmetry is the line $x = 0$.

Example 7: *Sketching a parabola*

QUESTION

Sketch the graph of $g(x) = -\frac{1}{2}x^2 - 3$. Mark the intercepts and the turning point.

SOLUTION

Step 1 : ***Examine the standard form of the equation***

We notice that $a < 0$. Therefore the graph is a "frown" and has a maximum turning point.

Step 2 : ***Calculate the intercepts***

For the y-intercept, let $x = 0$:

$$\begin{aligned} g(x) &= -\tfrac{1}{2}x^2 - 3 \\ g(0) &= -\tfrac{1}{2}(0)^2 - 3 \\ &= -3 \end{aligned}$$

This gives the point $(0; -3)$.

For the x-intercept let $y = 0$:

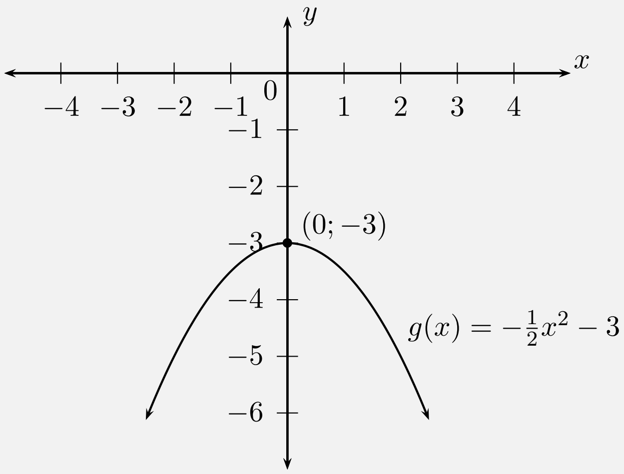
$$\begin{aligned} 0 &= -\tfrac{1}{2}x^2 - 3 \\ 3 &= -\tfrac{1}{2}x^2 \\ -2(3) &= x^2 \\ -6 &= x^2 \end{aligned}$$

There is no real solution, therefore there are no x-intercepts.

Step 3 : ***Determine the turning point***

From the standard form of the equation we see that the turning point is $(0; -3)$.

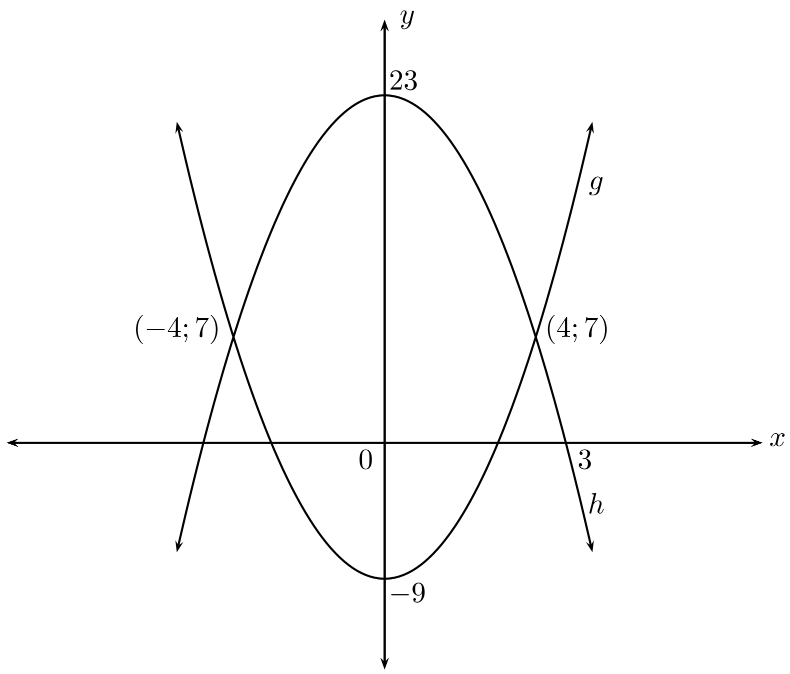
Step 4 : ***Plot the points and sketch the graph***



Domain: $x \in \mathbb{R}$.
Range: $y \in (-\infty; -3]$.
The axis of symmetry is the line $x = 0$.

Exercise 5 - 3

1. Show that if $a < 0$ the range of $f(x) = ax^2 + q$ is $\{f(x) : f(x) \leq q\}$.
2. Draw the graph of the function $y = -x^2 + 4$ showing all intercepts with the axes.
3. Two parabolas are drawn: $g : y = ax^2 + p$ and $h : y = bx^2 + q$.



(a) Find the values of a and p.

(b) Find the values of b and q.

(c) Find the values of x for which $g(x) \geq h(x)$.

(d) For what values of x is g increasing?

More practice video solutions or help at www.everythingmaths.co.za

(1.) 00fm (2.) 00fn (3a-d.) 00fp

5.4 Hyperbolic functions EMABF

Video: VMaxw at www.everythingmaths.co.za

Functions of the form $y = \frac{1}{x}$ EMABG

Functions of the general form $y = \frac{a}{x} + q$ are called hyperbolic functions.

Example 8: *Plotting a hyperbolic function*

QUESTION

$$y = h(x) = \frac{1}{x}$$

Complete the following table for $h(x) = \frac{1}{x}$ and plot the points on a system of axes.

x	-3	-2	-1	$-\frac{1}{2}$	$-\frac{1}{4}$	0	$\frac{1}{4}$	$\frac{1}{2}$	1	2	3
$h(x)$	$-\frac{1}{3}$										

1. *Join the points with smooth curves.*
2. *What happens if $x = 0$?*
3. *Explain why the graph consists of two separate curves.*
4. *What happens to $h(x)$ as the value of x becomes very small or very large?*
5. *The domain of $h(x)$ is $\{x : x \in \mathbb{R}, x \neq 0\}$. Determine the range.*
6. *About which two lines is the graph symmetrical?*

SOLUTION

Step 1 : ***Substitute values into the equation***

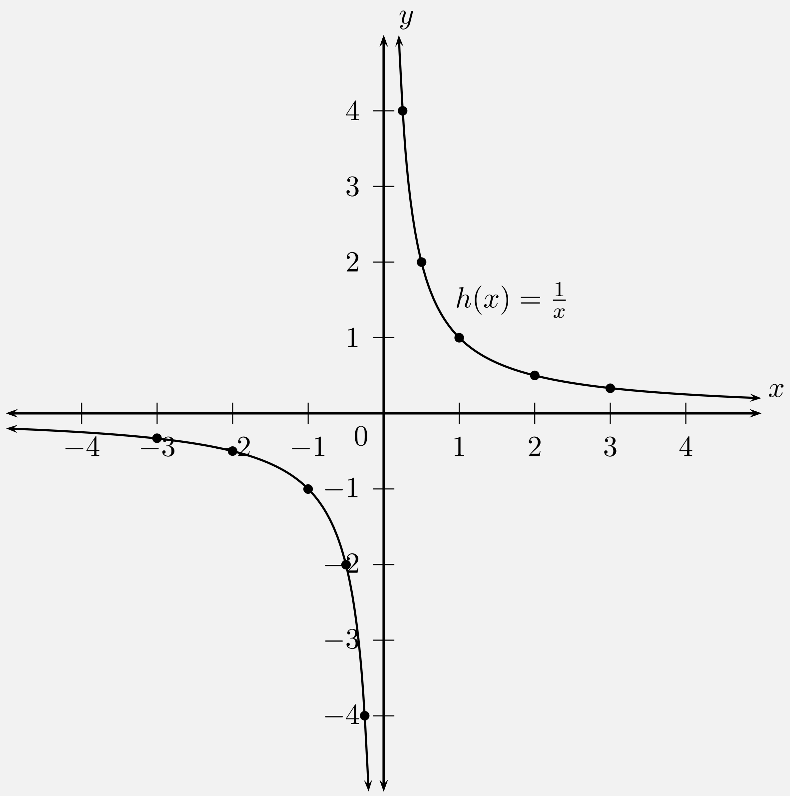
$$\begin{aligned}
h(x) &= \tfrac{1}{x} \\
h(-3) &= \tfrac{1}{-3} &&= -\tfrac{1}{3} \\
h(-2) &= \tfrac{1}{-2} &&= -\tfrac{1}{2} \\
h(-1) &= \tfrac{1}{-1} &&= -1 \\
h(-\tfrac{1}{2}) &= \frac{1}{-\frac{1}{2}} &&= -2 \\
h(-\tfrac{1}{4}) &= \frac{1}{-\frac{1}{4}} &&= -4 \\
h(0) &= \tfrac{1}{0} &&= \text{undefined}
\end{aligned}$$

$$h(\tfrac{1}{4}) = \frac{1}{\frac{1}{4}} = 4$$
$$h(\tfrac{1}{2}) = \frac{1}{\frac{1}{2}} = 2$$
$$h(1) = \tfrac{1}{1} = 1$$
$$h(2) = \tfrac{1}{2} = \tfrac{1}{2}$$
$$h(3) = \tfrac{1}{3} = \tfrac{1}{3}$$

x	-3	-2	-1	$-\frac{1}{2}$	$-\frac{1}{4}$	0	$\frac{1}{4}$	$\frac{1}{2}$	1	2	3
$h(x)$	$-\frac{1}{3}$	$-\frac{1}{2}$	-1	-2	-4	undefined	4	2	1	$\frac{1}{2}$	$\frac{1}{3}$

Step 2 : ***Plot the points and join with two smooth curves***

From the table we get the following points: $(-3;-\frac{1}{3})$, $(-2;-\frac{1}{2})$, $(-1;-1)$, $(-\frac{1}{2};-2)$, $(-\frac{1}{4};-4)$, $(\frac{1}{4};4)$, $(\frac{1}{2};2)$, $(1;1)$, $(2;\frac{1}{2})$, $(3;\frac{1}{3})$.



For $x = 0$ the function h is undefined. This is called a discontinuity at $x = 0$.

$y = h(x) = \frac{1}{x}$ therefore we can write that $x \times y = 1$. Since the product of two positive numbers **and** the product of two negative numbers can be equal to 1, the graph lies in the first and third quadrants.

Step 3 : ***Determine the asymptotes***

As the value of x gets larger, the value of $h(x)$ gets closer to, but does not equal 0. This is a horizontal asymptote, the line $y = 0$. The same happens in the third quadrant; as x gets smaller, $h(x)$ also approaches the negative x-axis asymptotically.

We also notice that there is a vertical asymptote, the line $x = 0$; as x gets closer to 0, $h(x)$ approaches the y-axis asymptotically.

Step 4 : ***Determine the range***

Domain: $\{x : x \in \mathbb{R}, x \neq 0\}$
From the graph, we see that y is defined for all values except 0.
Range: $\{y : y \in \mathbb{R}, y \neq 0\}$

Step 5 : ***Determine the lines of symmetry***

The graph of $h(x)$ has two axes of symmetry: the lines $y = x$ and $y = -x$. About these two lines, one half of the hyperbola is a mirror image of the other half.

Functions of the form $y = \frac{a}{x} + q$

EMABH

Investigation:	*The effects of a and q on a hyperbola*

On the same set of axes, plot the following graphs:

1. $y_1 = \dfrac{1}{x} - 2$
2. $y_2 = \dfrac{1}{x} - 1$
3. $y_3 = \dfrac{1}{x}$
4. $y_4 = \dfrac{1}{x} + 1$
5. $y_5 = \dfrac{1}{x} + 2$

Use your results to deduce the effect of q.

On the same set of axes, plot the following graphs:

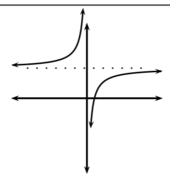
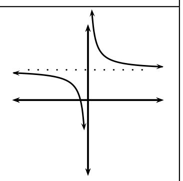
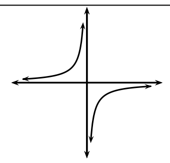
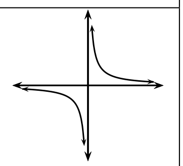
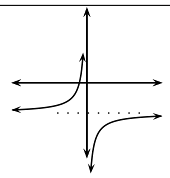
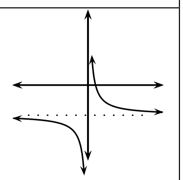
6. $y_6 = \frac{-2}{x}$

7. $y_7 = \frac{-1}{x}$

8. $y_8 = \frac{1}{x}$

9. $y_9 = \frac{2}{x}$

Use your results to deduce the effect of a.

	$a < 0$	$a > 0$
$q > 0$		
$q = 0$		
$q < 0$		

The effect of q

The effect of q is called a vertical shift because all points are moved the same distance in the same direction (it slides the entire graph up or down).

- For $q > 0$, the graph of $f(x)$ is shifted vertically upwards by q units.
- For $q < 0$, the graph of $f(x)$ is shifted vertically downwards by q units.

The horizontal asymptote is the line $y = q$ and the vertical asymptote is always the y-axis, the line $x = 0$.

The effect of a

The sign of a determines the shape of the graph.

- If $a > 0$, the graph of $f(x)$ lies in the first and third quadrants.
  For $a > 1$, the graph of $f(x)$ will be further away from the axes than $y = \frac{1}{x}$.
  For $0 < a < 1$, as a tends to 0, the graph moves closer to the axes than $y = \frac{1}{x}$.
- If $a < 0$, the graph of $f(x)$ lies in the second and fourth quadrants.
  For $a < -1$, the graph of $f(x)$ will be further away from the axes than $y = -\frac{1}{x}$.
  For $-1 < a < 0$, as a tends to 0, the graph moves closer to the axes than $y = -\frac{1}{x}$.

Discovering the characteristics EMABI

The standard form of a hyperbola is the equation $y = \frac{a}{x} + q$.

Domain and range

For $y = \frac{a}{x} + q$, the function is undefined for $x = 0$. The domain is therefore $\{x : x \in \mathbb{R}, x \neq 0\}$.

We see that $y = \frac{a}{x} + q$ can be re-written as:

$$
\begin{aligned}
y &= \frac{a}{x} + q \\
y - q &= \frac{a}{x} \\
\text{If } x \neq 0 \text{ then : } (y-q)x &= a \\
x &= \frac{a}{y-q}
\end{aligned}
$$

This shows that the function is undefined only at $y = q$.

Therefore the range is $\{f(x) : f(x) \in \mathbb{R}, f(x) \neq q\}$.

Example 9: *Domain and range of a hyperbola*

QUESTION

If $g(x) = \frac{2}{x} + 2$, determine the domain and range of the function.

SOLUTION

Step 1 : ***Determine the domain***

The domain is $\{x : x \in \mathbb{R}, x \neq 0\}$ because $g(x)$ is undefined only at $x = 0$.

Step 2 : ***Determine the range***

We see that $g(x)$ is undefined only at $y = 2$. Therefore the range is $\{g(x) : g(x) \in \mathbb{R}, g(x) \neq 2\}$.

Intercepts

The y-intercept:
Every point on the y-axis has an x-coordinate of 0, therefore to calculate the y-intercept, let $x = 0$.
For example, the y-intercept of $g(x) = \frac{2}{x} + 2$ is given by setting $x = 0$:

$$\begin{aligned} y &= \frac{2}{x} + 2 \\ y &= \frac{2}{0} + 2 \end{aligned}$$

which is undefined, therefore there is no y-intercept.

The x-intercept:
Every point on the x-axis has a y-coordinate of 0, therefore to calculate the x-intercept, let $y = 0$.

For example, the x-intercept of $g(x) = \frac{2}{x} + 2$ is given by setting $y = 0$:

$$\begin{aligned}
y &= \frac{2}{x} + 2 \\
0 &= \frac{2}{x} + 2 \\
\frac{2}{x} &= -2 \\
x &= \frac{2}{-2} \\
&= -1
\end{aligned}$$

This gives the point $(-1; 0)$.

Asymptotes

There are two asymptotes for functions of the form $y = \frac{a}{x} + q$.

The horizontal asymptote is the line $y = q$ and the vertical asymptote is always the y-axis, the line $x = 0$.

Axes of symmetry

There are two lines about which a hyperbola is symmetrical: $y = x + q$ and $y = -x + q$.

Sketching graphs of the form $y = \frac{a}{x} + q$ EMABJ

In order to sketch graphs of functions of the form, $y = f(x) = \frac{a}{x} + q$, we need to determine four characteristics:

1. sign of a
2. y-intercept
3. x-intercept
4. asymptotes

Example 10: *Sketching a hyperbola*

QUESTION

Sketch the graph of $g(x) = \frac{2}{x} + 2$. Mark the intercepts and the asymptotes.

SOLUTION

Step 1 : ***Examine the standard form of the equation***

We notice that $a > 0$ therefore the graph of $g(x)$ lies in the first and third quadrant.

Step 2 : ***Calculate the intercepts***

For the y-intercept, let $x = 0$:

$$\begin{aligned} g(x) = &\ \frac{2}{x} + 2 \\ g(0) = &\ \frac{2}{0} + 2 \end{aligned}$$

This is undefined, therefore there is no y-intercept.

For the x-intercept, let $y = 0$:

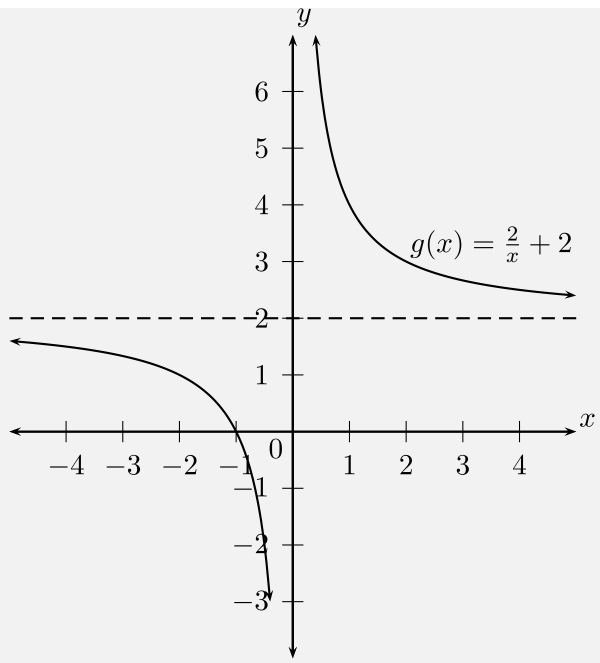
$$\begin{aligned} g(x) &= \frac{2}{x} + 2 \\ 0 &= \frac{2}{x} + 2 \\ \frac{2}{x} &= -2 \\ \therefore x &= -1 \end{aligned}$$

This gives the point $(-1; 0)$.

Step 3 : ***Determine the asymptotes***

The horizontal asymptote is the line $y = 2$. The vertical asymptote is the line $x = 0$.

Step 4 : ***Sketch the graph***



Domain: $\{x : x \in \mathbb{R}, x \neq 0\}$.
Range: $\{y : y \in \mathbb{R}, y \neq 2\}$.

Example 11: *Sketching a hyperbola*

QUESTION

Sketch the graph of $y = \frac{-4}{x} + 7$.

SOLUTION

Step 1 : ***Examine the standard form of the equation***

We see that $a < 0$ therefore the graph lies in the second and fourth quadrants.

Step 2 : ***Calculate the intercepts***

For the y-intercept, let $x = 0$:

$$\begin{aligned} y &= \frac{-4}{x} + 7 \\ &= \frac{-4}{0} + 7 \end{aligned}$$

This is undefined, therefore there is no y-intercept.

For the x-intercept, let $y = 0$:

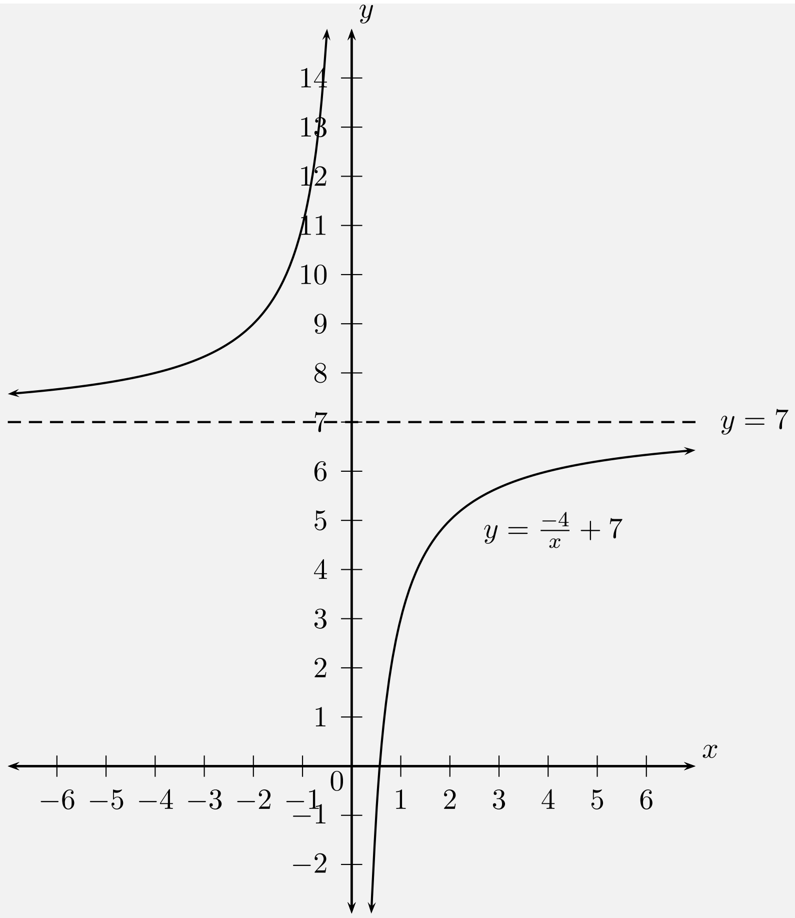
$$\begin{aligned} y &= \frac{-4}{x} + 7 \\ 0 &= \frac{-4}{x} + 7 \\ \frac{-4}{x} &= -7 \\ \therefore x &= \frac{4}{7} \end{aligned}$$

This gives the point $\left(\frac{4}{7}; 0\right)$.

Step 3 : ***Determine the asymptotes***

The horizontal asymptote is the line $y = 7$. The vertical asymptote is the line $x = 0$.

Step 4 : ***Sketch the graph***



Domain: $\{x : x \in \mathbb{R}, x \neq 0\}$
Range: $\{y : y \in \mathbb{R}, y \neq 7\}$
Axis of symmetry: $y = x + 7$ and $y = -x + 7$

Exercise 5 - 4

1. Draw the graph of $xy = -6$.
   - **(a)** Does the point $(-2; 3)$ lie on the graph? Give a reason for your answer.
   - **(b)** If the x-value of a point on the drawn graph is $0{,}25$ what is the corresponding y-value?
   - **(c)** What happens to the y-values as the x-values become very large?
   - **(d)** Give the equations of the asymptotes.
   - **(e)** With the line $y = -x$ as line of symmetry, what is the point symmetrical to $(-2; 3)$?

2. Draw the graph of $h(x) = \frac{8}{x}$.
   (a) How would the graph $g(x) = \frac{8}{x} + 3$ compare with that of $h(x) = \frac{8}{x}$? Explain your answer fully.
   (b) Draw the graph of $y = \frac{8}{x} + 3$ on the same set of axes, showing asymptotes, axes of symmetry and the coordinates of one point on the graph.

More practice video solutions or help at www.everythingmaths.co.za

(1a-e.) 00fq (2a-b.) 00fr

5.5 Exponential functions EMABK

Functions of the form $y = b^x$ EMABL

Functions of the general form $y = ab^x + q$ are called exponential functions. In the equation a and q are constants and have different effects on the function.

Video: VMaxx at www.everythingmaths.co.za

Example 12: *Plotting an exponential function*

QUESTION

$$y = f(x) = b^x \text{ for } b > 0 \text{ and } b \neq 1$$

Complete the following table for each of the functions and draw the graphs on the same system of axes: $f(x) = 2^x$, $g(x) = 3^x$, $h(x) = 5^x$.

	-2	-1	0	1	2
$f(x) = 2^x$					
$g(x) = 3^x$					
$h(x) = 5^x$					

1. *At what point do these graphs intersect?*
2. *Explain why they do not cut the x-axis.*
3. *Give the domain and range of $h(x)$.*
4. *As x increases, does $h(x)$ increase or decrease?*
5. *Which of these graphs increases at the slowest rate?*
6. *For $y = k^x$ and $k > 1$, the greater the value of k, the steeper the curve of the graph. True or false?*

Complete the following table for each of the functions and draw the graphs on the same system of axes: $F(x) = (\frac{1}{2})^x$, $G(x) = (\frac{1}{3})^x$, $H(x) = (\frac{1}{5})^x$.

	-2	-1	0	1	2
$F(x) = (\frac{1}{2})^x$					
$G(x) = (\frac{1}{3})^x$					
$H(x) = (\frac{1}{5})^x$					

7. *Give the y-intercept for each function.*
8. *Describe the relationship between the graphs $f(x)$ and $F(x)$.*
9. *Describe the relationship between the graphs $g(x)$ and $G(x)$.*
10. *Give the domain and range of $H(x)$.*
11. *For $y = (\frac{1}{k})^x$ and $k > 1$, the greater the value of k, the steeper the curve of the graph. True or false?*
12. *Give the equation of the asymptote for the functions.*

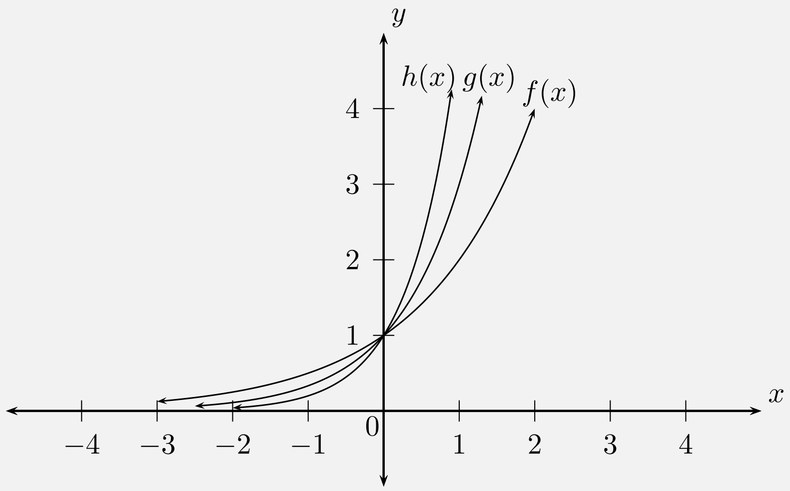
SOLUTION

Step 1 : ***Substitute values into the equations***

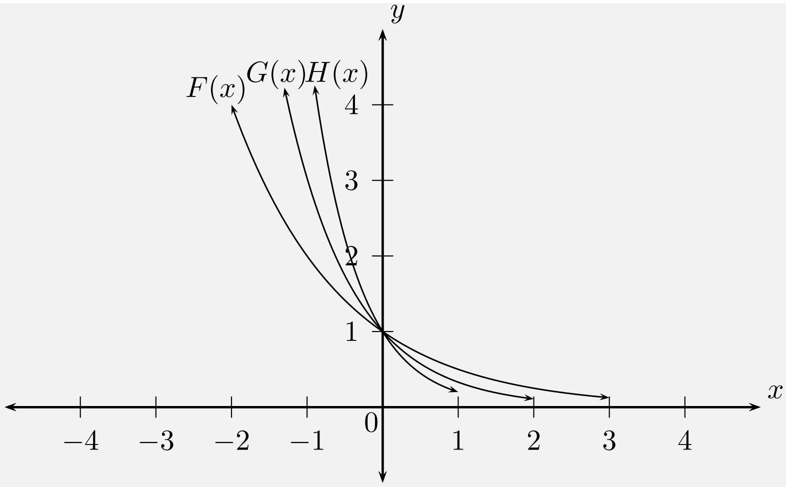
	-2	-1	0	1	2
$f(x) = 2^x$	$\frac{1}{4}$	$\frac{1}{2}$	1	2	4
$g(x) = 3^x$	$\frac{1}{9}$	$\frac{1}{3}$	1	3	9
$g(x) = 5^x$	$\frac{1}{25}$	$\frac{1}{5}$	1	5	25

	-2	-1	0	1	2
$F(x) = (\frac{1}{2})^x$	4	2	1	$\frac{1}{2}$	$\frac{1}{4}$
$G(x) = (\frac{1}{3})^x$	9	3	1	$\frac{1}{3}$	$\frac{1}{9}$
$H(x) = (\frac{1}{5})^x$	25	5	1	$\frac{1}{5}$	$\frac{1}{25}$

Step 2 : ***Plot the points and join with a smooth curve***



1. We notice that all graphs pass through the point $(0; 1)$. Any number with exponent 0 is equal to 1.
2. The graphs do not cut the x-axis because 0^0 is undefined.
3. Domain: $\{x : x \in \mathbb{R}\}$.
   Range: $\{y : y \in \mathbb{R},\ y > 0\}$.
4. As x increases, $h(x)$ increases.
5. $f(x) = 2^x$ increases at the slowest rate because it has the smallest base.
6. True: the greater the value of k $(k > 1)$, the steeper the graph of $y = k^x$.



1. The y-intercept is the point $(0;1)$ for all graphs. For any real number z, $z^0 = 1$.
2. $F(x)$ is the reflection of $f(x)$ about the y-axis.
3. $G(x)$ is the reflection of $g(x)$ about the y-axis.
4. Domain: $\{x : x \in \mathbb{R}\}$.
   Range: $\{y : y \in \mathbb{R},\ y > 0\}$.
5. True: the greater the value of k $(k > 1)$, the steeper the graph of $y = (\frac{1}{k})^x$.
6. The equation of the horizontal asymptote is $y = 0$, the x-axis.

Functions of the form $y = ab^x + q$ EMABM

Investigation:	*The effects of a and q on an exponential graph*

On the same set of axes, plot the following graphs ($b = 2$, $a = 1$ and q changes):

1. $y_1 = 2^x - 2$
2. $y_2 = 2^x - 1$
3. $y_3 = 2^x$
4. $y_4 = 2^x + 1$
5. $y_5 = 2^x + 2$

	-2	-1	0	1	2
$y_1 = 2^x - 2$					
$y_2 = 2^x - 1$					
$y_3 = 2^x$					
$y_4 = 2^x + 1$					
$y_5 = 2^x + 2$					

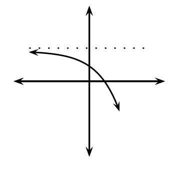
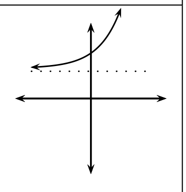
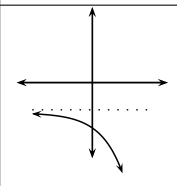
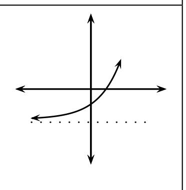
Use your results to deduce the effect of q.

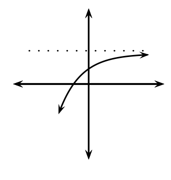
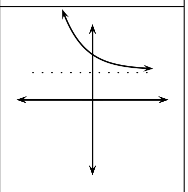
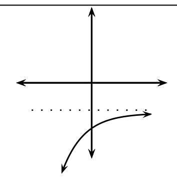
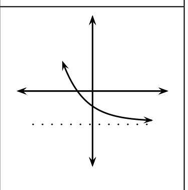
On the same set of axes, plot the following graphs ($b = 2$, $q = 0$ and a changes):

6. $y_6 = 2^x$
7. $y_7 = 2 \times 2^x$
8. $y_8 = -2^x$
9. $y_9 = -2 \times 2^x$

	-2	-1	0	1	2
$y_6 = 2^x$					
$y_7 = 2 \times 2^x$					
$y_8 = -2^x$					
$y_9 = -2 \times 2^x$					

Use your results to deduce the effect of a.

$\boxed{b > 1}$	$a < 0$	$a > 0$
$q > 0$		
$q < 0$		

$\boxed{0 < b < 1}$	$a < 0$	$a > 0$
$q > 0$		
$q < 0$		

The effect of q

The effect of q is called a vertical shift because all points are moved the same distance in the same direction (it slides the entire graph up or down).

- For $q > 0$, the graph is shifted vertically upwards by q units.
- For $q < 0$, the graph is shifted vertically downwards by q units.

The horizontal asymptote is shifted by q units and is the line $y = q$.

The effect of a

The sign of a determines whether the graph curves upwards or downwards.

- For $a > 0$, the graph curves upwards.
- For $a < 0$, the graph curves downwards. It reflects the graph about the horizontal asymptote.

Discovering the characteristics

EMABN

The standard form of an exponential function is $y = ab^x + q$.

Domain and range

For $y = ab^x + q$, the function is defined for all real values of x. Therefore, the domain is $\{x : x \in \mathbb{R}\}$.

The range of $y = ab^x + q$ is dependent on the sign of a.

For $a > 0$:

$$\begin{aligned} b^x &> 0 \\ ab^x &> 0 \\ ab^x + q &> q \\ f(x) &> q \end{aligned}$$

Therefore, for $a > 0$ the range is $\{f(x) : f(x) > q\}$.

For $a < 0$:

$$\begin{aligned} b^x &> 0 \\ ab^x &< 0 \\ ab^x + q &< q \\ f(x) &< q \end{aligned}$$

Therefore, for $a < 0$ the range is $\{f(x) : f(x) < q\}$.

Example 13: *Domain and range of an exponential function*

QUESTION

Find the domain and range of $g(x) = 5.2^x + 1$

SOLUTION

Step 1 : ***Find the domain***

The domain of $g(x) = 5 \times 2^x + 1$ is $\{x : x \in \mathbb{R}\}$.

Step 2 : ***Find the range***

$$\begin{aligned} 2^x &> 0 \\ 5 \times 2^x &> 0 \\ 5 \times 2^x + 1 &> 1 \end{aligned}$$

Therefore the range is $\{g(x) : g(x) > 1\}$.

Intercepts

The y-intercept:

For the y-intercept, let $x = 0$:

$$\begin{aligned} y &= ab^x + q \\ &= ab^0 + q \\ &= a(1) + q \\ &= a + q \end{aligned}$$

For example, the y-intercept of $g(x) = 5 \times 2^x + 1$ is given by setting $x = 0$:

$$\begin{aligned} y &= 5 \times 2^x + 1 \\ &= 5 \times 2^0 + 1 \\ &= 5 + 1 \\ &= 6 \end{aligned}$$

This gives the point $(0; 6)$.

The x-intercept:
For the x-intercept, let $y = 0$.
For example, the x-intercept of $g(x) = 5 \times 2^x + 1$ is given by setting $y = 0$:

$$\begin{aligned} y &= 5 \times 2^x + 1 \\ 0 &= 5 \times 2^x + 1 \\ -1 &= 5 \times 2^x \\ 2^x &= -\frac{1}{5} \end{aligned}$$

There is no real solution. Therefore, the graph of $g(x)$ does not have any x-intercepts.

Asymptotes

Exponential functions of the form $y = ab^x + q$ have a single horizontal asymptote, the line $x = q$.

Sketching graphs of the form $y = ab^x + q$ EMABO

In order to sketch graphs of functions of the form, $y = ab^x + q$, we need to determine four characteristics:

1. sign of a
2. y-intercept
3. x-intercept
4. asymptote

Example 14: *Sketching an exponential function*

QUESTION

Sketch the graph of $g(x) = 3 \times 2^x + 2$. *Mark the intercept and the asymptote.*

SOLUTION

Step 1 : ***Examine the standard form of the equation***

From the equation we see that $a > 1$, therefore the graph curves upwards. $q > 0$ therefore the graph is shifted vertically upwards by 2 units.

Step 2 : ***Calculate the intercepts***

For the y-intercept, let $x = 0$:

$$\begin{aligned} y &= 3 \times 2^x + 2 \\ &= 3 \times 2^0 + 2 \\ &= 3 + 2 \\ &= 5 \end{aligned}$$

This gives the point $(0;5)$.
For the x-intercept, let $y = 0$:

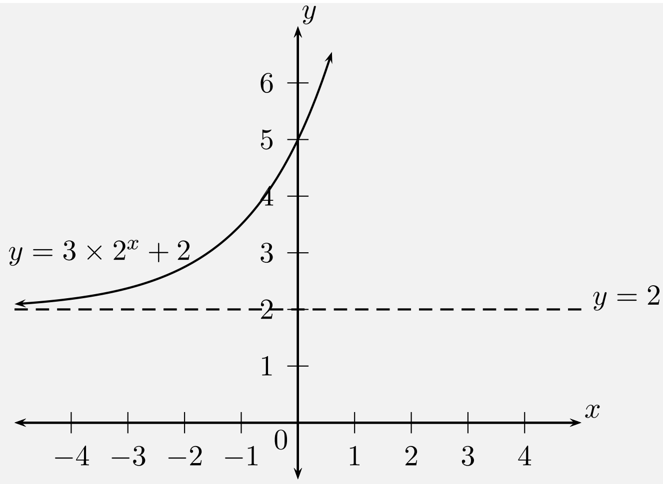
$$\begin{aligned} y &= 3 \times 2^x + 2 \\ 0 &= 3 \times 2^x + 2 \\ -2 &= 3 \times 2^x \\ 2^x &= -\frac{2}{3} \end{aligned}$$

There is no real solution, therefore there is no x-intercept.

Step 3 : ***Determine the asymptote***

The horizontal asymptote is the line $y = 2$.

Step 4 : ***Plot the points and sketch the graph***



Domain: $\{x : x \in \mathbb{R}\}$
Range: $\{g(x) : g(x) > 2\}$
Note that there is no axis of symmetry for exponential functions.

Example 15: *Sketching an exponential graph*

QUESTION

Sketch the graph of $y = -2 \times 3^x + 6$.

SOLUTION

Step 1 : ***Examine the standard form of the equation***

From the equation we see that $a < 0$ therefore the graph curves downwards. $q > 0$ therefore the graph is shifted vertically upwards by 6 units.

Step 2 : ***Calculate the intercepts***

For the y-intercept, let $x = 0$:

$$\begin{aligned} y &= -2 \times 3^x + 6 \\ &= -2 \times 3^0 + 6 \\ &= 4 \end{aligned}$$

This gives the point $(0;4)$.
For the x-intercept, let $y=0$:

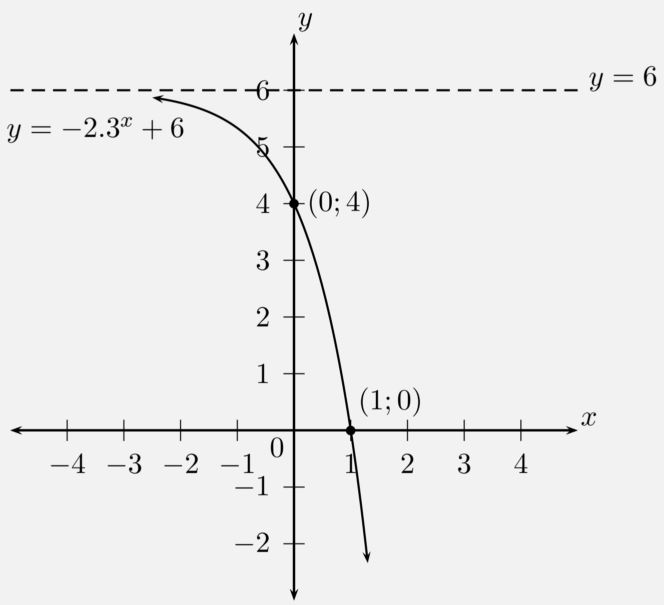
$$\begin{aligned} y &= -2\times 3^x+6 \\ 0 &= -2\times 3^x+6 \\ -6 &= -2\times 3^x \\ 3^1 &= 3^x \\ \therefore x &= 1 \end{aligned}$$

This gives the point $(1;0)$.

Step 3 : ***Determine the asymptote***

The horizontal asymptote is the line $y=6$.

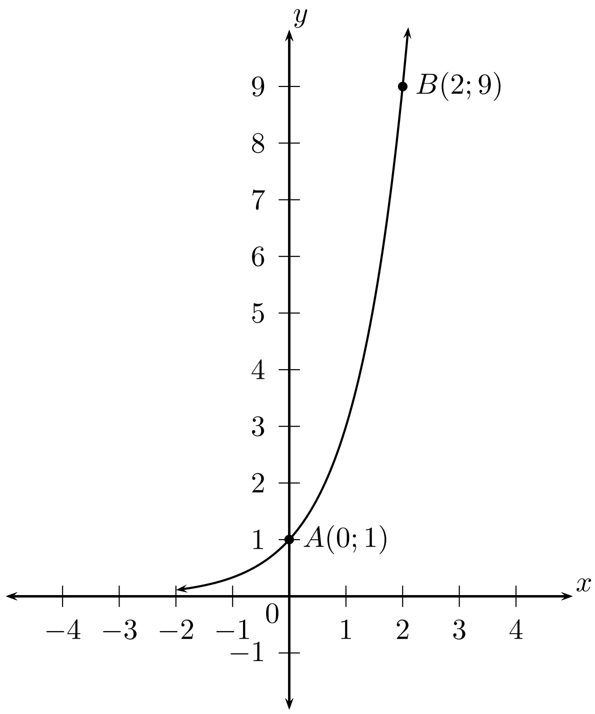
Step 4 : ***Plot the points and sketch the graph***



Domain: $\{x : x\in\mathbb{R}\}$
Range: $\{g(x) : g(x)<6\}$

Exercise 5 - 5

1. Draw the graphs of $y = 2^x$ and $y = (\frac{1}{2})^x$ on the same set of axes.
   (a) Is the x-axis an asymptote or an axis of symmetry to both graphs? Explain your answer.
   (b) Which graph is represented by the equation $y = 2^{-x}$? Explain your answer.
   (c) Solve the equation $2^x = (\frac{1}{2})^x$ graphically and check that your answer is correct by using substitution.
2. The curve of the exponential function f in the accompanying diagram cuts the y-axis at the point $A(0; 1)$ and passes through the point $B(2; 9)$.



   (a) Determine the equation of the function f.
   (b) Determine the equation of h, the reflection of f in the x-axis.
   (c) Determine the range of h.
   (d) Determine the equation of g, the reflection of f in the y-axis.
   (e) Determine the equation of j if j is a vertical stretch of f by $+2$ units.
   (f) Determine the equation of k if k is a vertical shift of f by -3 units.

More practice video solutions or help at www.everythingmaths.co.za

(1a-c.) 00fs (2a-f.) 00ft

5.6 Trigonometric functions EMABP

This section describes the graphs of trigonometric functions.

Video: VMazc at www.everythingmaths.co.za

Sine function EMABQ

Functions of the form $y = sin\ \theta$ EMABR

Example 16: *Plotting a sine graph*

QUESTION

$$y = f(\theta) = sin\ \theta \qquad [0° \leq \theta \leq 360°]$$

Use your calculator to complete the following table.
Choose an appropriate scale and plot the values of θ on the x-axis and of $sin\ \theta$ on the y-axis. (Round answers to 2 decimal places).

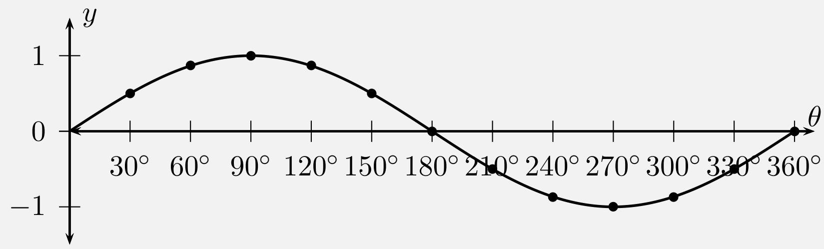
θ	$0°$	$30°$	$60°$	$90°$	$120°$	$150°$	$180°$	$210°$	$240°$	$270°$	$300°$	$330°$	$360°$
$sin\ \theta$													

SOLUTION

Step 1 : ***Substitute values for θ***

θ	$0°$	$30°$	$60°$	$90°$	$120°$	$150°$	$180°$	$210°$	$240°$	$270°$	$300°$	$330°$	$360°$
$sin\ \theta$	0	$0{,}5$	$0{,}87$	1	$0{,}87$	$0{,}5$	0	$-0{,}5$	$-0{,}87$	-1	$-0{,}87$	$-0{,}5$	0

Step 2 : ***Plot the points and join with a smooth curve***



Notice the wave shape of the graph. Each complete wave takes $360°$ to complete. This is called the period. The height of the wave above and below the x-axis is called the graph's amplitude. The maximum value of $y = sin\ \theta$ is 1 and the minimum value is -1.

Domain: $[\ 0°; 360°]$
Range: $[-1; 1]$
x-intercepts: $(0°; 0)$, $(180°; 0)$, $(360°; 0)$
y-intercept: $(0°; 0)$
Maximum turning point: $(90°; 1)$
Minimum turning point: $(270°; -1)$

Functions of the form $y = a\ sin\ \theta + q$ EMABS

Investigation:	*The effects of a and q on a sine graph*

In the equation, $y = a\ sin\ \theta + q$, a and q are constants and have different effects on the graph. On the same set of axes, plot the following graphs for $0° \leq \theta \leq 360°$:

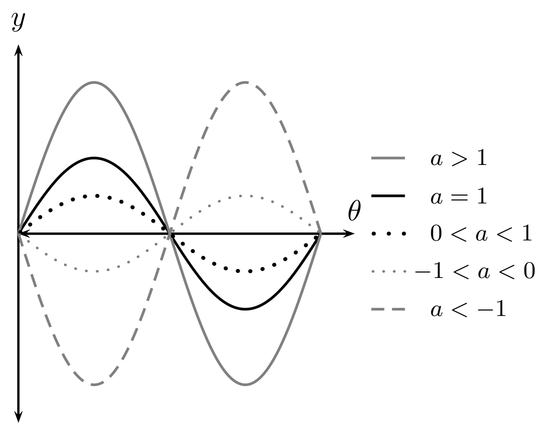
1. $y_1 = sin\ \theta - 2$
2. $y_2 = sin\ \theta - 1$
3. $y_3 = sin\ \theta$
4. $y_4 = sin\ \theta + 1$
5. $y_5 = sin\ \theta + 2$

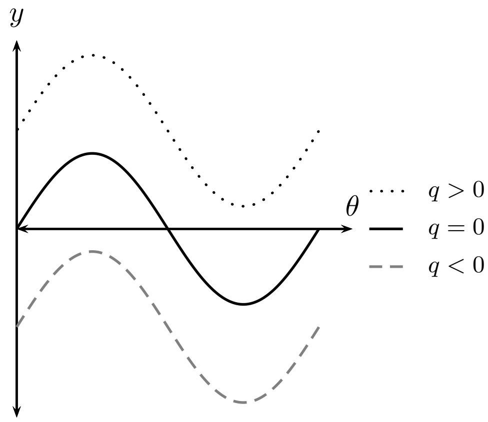
Use your results to deduce the effect of q.

On the same set of axes, plot the following graphs for $0° \leq \theta \leq 360°$:

6. $y_6 = -2\ sin\ \theta$
7. $y_7 = -sin\ \theta$
8. $y_8 = sin\ \theta$
9. $y_9 = 2\ sin\ \theta$

Use your results to deduce the effect of a.

Effect of a	
$a > 1$: vertical stretch, amplitude increases	
$a = 1$: basic sine graph	
$0 < a < 1$: vertical contraction, amplitude decreases	
$-1 < a < 0$: reflection about x-axis of $0 < a < 1$	
$a < -1$: reflection about x-axis of $a > 1$	

Effect of q	
$q > 0$: vertical shift upwards by q units	
$q = 0$: basic sine graph	
$q < 0$: vertical shift downwards by q units	

The effect of q
The effect of q is called a vertical shift because the whole sine graph shifts up or down by q units.

- For $q > 0$, the graph is shifted vertically upwards by q units.
- For $q < 0$, the graph is shifted vertically downwards by q units.

The effect of a
The value of a affects the amplitude of the graph; the height of the peaks and the depth of the troughs.

- For $a > 1$, there is a vertical stretch and the amplitude increases.
  For $0 < a < 1$, the amplitude decreases.
- For $a < 0$, there is a reflection about the x-axis.
  For $-1 < a < 0$, there is a reflection about the x-axis and the amplitude decreases.
  For $a < -1$, there is a reflection about the x-axis and the amplitude increases.

Note that amplitude is always positive.

Discovering the characteristics

EMABT

Domain and range

For $f(\theta) = a\ sin\ \theta + q$, the domain is $[\ 0°; 360°]$.

The range of $f(\theta) = a\ sin\ \theta + q$ depends on the values for a and q:

For $a > 0$:

$$\begin{array}{rcccl} -1 & \leq & sin\ \theta & \leq & 1 \\ -a & \leq & a\ sin\ \theta & \leq & a \\ -a+q & \leq & a\ sin\ \theta + q & \leq & a+q \\ -a+q & \leq & f(\theta) & \leq & a+q \end{array}$$

For all values of θ, $f(\theta)$ is always between $-a+q$ and $a+q$.
Therefore for $a > 0$, the range of $f(\theta) = a\ sin\ \theta + q$ is $\{f(\theta) : f(\theta) \in [-a+q, a+q]\}$.
Similarly, for $a < 0$, the range of $f(\theta) = a\ sin\ \theta + q$ is $\{f(\theta) : f(\theta) \in [a+q, -a+q]\}$.

Period

The period of $y = a\ sin\ \theta + q$ is $360°$. This means that one sine wave is completed in $360°$.

Intercepts

The y-intercept of $f(\theta) = a\ sin\ \theta + q$ is simply the value of $f(\theta)$ at $\theta = 0°$.

$$\begin{array}{rcl} y & = & f(0°) \\ & = & a\ sin\ 0° + q \\ & = & a(0) + q \\ & = & q \end{array}$$

This gives the point $(0; q)$.

Important: when sketching trigonometric graphs, always start with the basic graph and then consider the effects of a and q.

Example 17: *Sketching a sine graph*

QUESTION

Sketch the graph of $f(\theta) = 2\ sin\ \theta + 3$ for $\theta \in [\,0°; 360°]$.

SOLUTION

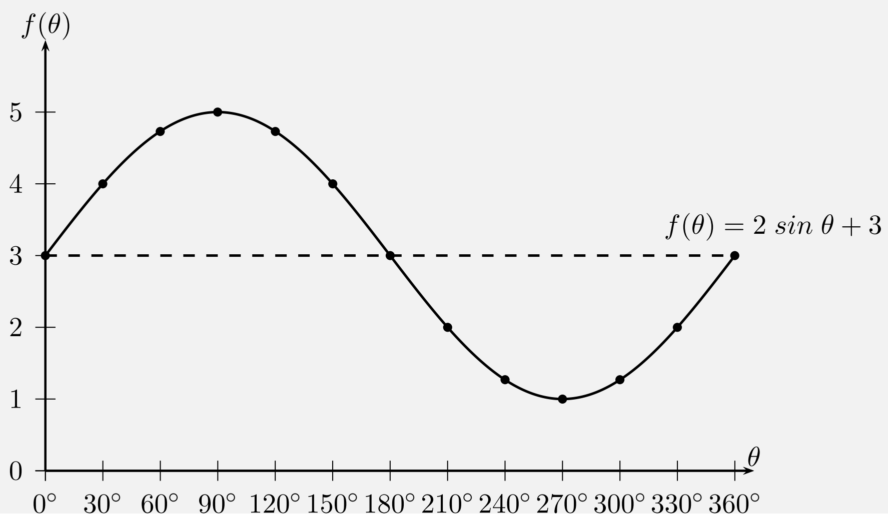
Step 1 : ***Examine the standard form of the equation***

From the equation we see that $a > 1$ so the graph is stretched vertically. We also see that $q > 0$ so the graph is shifted vertically upwards by 3 units.

Step 2 : ***Substitute values for θ***

θ	0°	30°	60°	90°	120°	150°	180°	210°	240°	270°	300°	330°	360°
$f(\theta)$	3	4	4,73	5	4,73	4	3	2	1,27	1	1,27	2	3

Step 3 : ***Plot the points and join with a smooth curve***



Domain: $[0°; 360°]$
Range: $[1; 5]$
x-intercepts: none
y-intercept: $(0°; 3)$
Maximum turning point: $(90°; 5)$
Minimum turning point: $(270°; 1)$

Cosine function

EMABU

Functions of the form $y = \cos\theta$

EMABV

Example 18: *Plotting a cosine graph*

QUESTION

$$y = f(\theta) = \cos\theta \qquad [0° \leq \theta \leq 360°]$$

Use your calculator to complete the following table.
Choose an appropriate scale and plot the values of θ on the x-axis and $\cos\theta$ on the y-axis. (Round answers to 2 decimal places.)

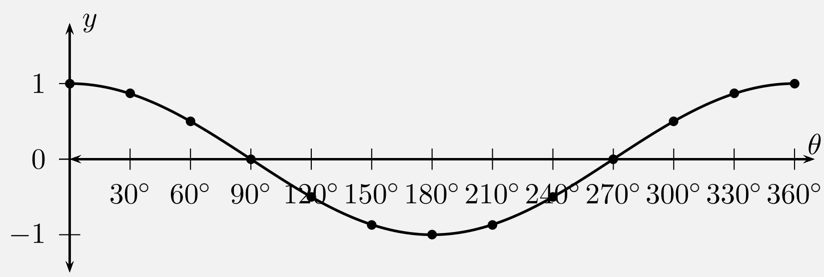
θ	$0°$	$30°$	$60°$	$90°$	$120°$	$150°$	$180°$	$210°$	$240°$	$270°$	$300°$	$330°$	$360°$
$\cos\theta$													

SOLUTION

Step 1 : ***Substitute values for*** θ

θ	$0°$	$30°$	$60°$	$90°$	$120°$	$150°$	$180°$	$210°$	$240°$	$270°$	$300°$	$330°$	$360°$
$cos\ \theta$	1	0.87	$0{,}5$	0	$-0{,}5$	$-0{,}87$	-1	$-0{,}87$	$-0{,}5$	0	$0{,}5$	$0{,}87$	1

Step 2 : ***Plot the points and join with a smooth curve***



Notice the similar wave shape of the graph. The period is also $360°$ and the amplitude is 1. The maximum value of $y = cos\ \theta$ is 1 and the minimum value is -1. Domain: $[\,0°; 360°]$
Range: $[-1; 1]$
x-intercepts: $(90°; 0)$, $(270°; 0)$
y-intercept: $(0°; 1)$
Maximum turning points: $(0°; 1)$, $(360°; 1)$
Minimum turning point: $(180°; -1)$

Functions of the form $y = a\ cos\ \theta + q$ EMABW

Investigation:	*The effects of a and q on a cosine graph*

In the equation, $y = a\ cos\ \theta + q$, a and q are constants and have different effects on the graph. On the same set of axes, plot the following graphs for $0° \leq \theta \leq 360°$:

1. $y_1 = cos\ \theta - 2$
2. $y_2 = cos\ \theta - 1$

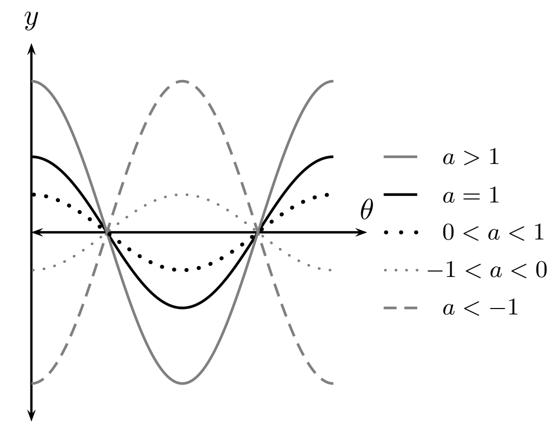
3. $y_3 = cos\ \theta$
4. $y_4 = cos\ \theta + 1$
5. $y_5 = cos\ \theta + 2$

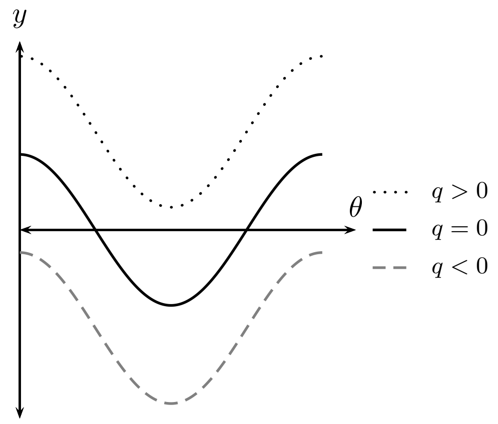
Use your results to deduce the effect of q.

On the same set of axes, plot the following graphs for $0° \leq \theta \leq 360°$:

6. $y_6 = -2cos\ \theta$
7. $y_7 = -cos\ \theta$
8. $y_8 = cos\ \theta$
9. $y_9 = 2cos\ \theta$

Use your results to deduce the effect of a.

Effect of a	
$a > 1$: vertical stretch, amplitude increases	
$a = 1$: basic cosine graph	
$0 < a < 1$: amplitude decreases	
$-1 < a < 0$: reflection about x-axis, amplitude decreases	
$a < -1$: reflection about x-axis, amplitude increases	

Effect of q	
$q > 0$: vertical shift upwards by q units	
$q = 0$: basic cosine graph	
$q < 0$: vertical shift downwards by q units	

The effect of q
The effect of q is called a vertical shift because the whole cosine graph shifts up or down by q units.

- For $q > 0$, the graph is shifted vertically upwards by q units.
- For $q < 0$, the graph is shifted vertically downwards by q units.

The effect of a
The value of a affects the amplitude of the graph; the height of the peaks and the depth of the troughs.

- For $a > 0$, there is a vertical stretch and the amplitude increases.
  For $0 < a < 1$, the amplitude decreases.
- For $a < 0$, there is a reflection about the x-axis.
  For $-1 < a < 0$, there is a reflection about the x-axis and the amplitude decreases.
  For $a < -1$, there is reflection about the x-axis and the amplitude increases.

Note that amplitude is always positive.

Discovering the characteristics

EMABX

Domain and range

For $f(\theta) = a\ cos\ \theta + q$, the domain is $[\ 0°; 360°]$.

It is easy to see that the range of $f(\theta)$ will be the same as the range of $a\ sin\ \theta + q$. This is because the maximum and minimum values of $a\ cos\ (\theta) + q$ will be the same as the maximum and minimum values of $a\ sin\ \theta + q$.

For $a > 0$ the range of $f(\theta) = a\ cos\ \theta + q$ is $\{f(\theta) : f(\theta) \in [-a + q; a + q]\}$.
For $a < 0$ the range of $f(\theta) = a\ cos\ \theta + q$ is $\{f(\theta) : f(\theta) \in [a + q; -a + q]\}$.

Period

The period of $y = a\ cos\ \theta + q$ is $360°$. This means that one cosine wave is completed in $360°$.

Intercepts

The y-intercept of $f(\theta) = a\ cos\ \theta + q$ is calculated in the same way as for sine.

$$\begin{aligned} y &= f(0°) \\ &= a\ cos\ 0° + q \\ &= a(1) + q \\ &= a + q \end{aligned}$$

This gives the point $(0°; a + q)$.

Example 19: *Sketching a cosine graph*

QUESTION

Sketch the graph of $f(\theta) = 2\ cos\ \theta + 3$ *for* $\theta \in [\ 0°; 360°]$.

SOLUTION

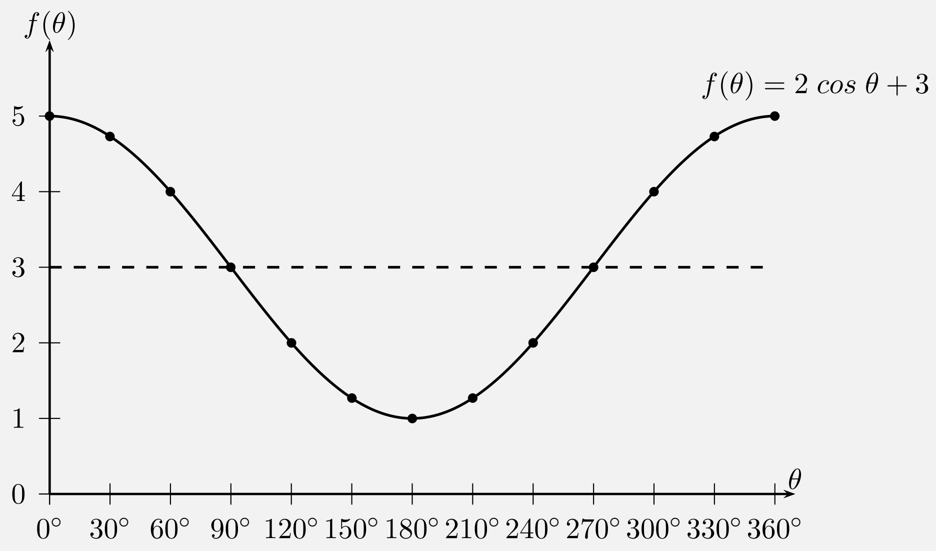
Step 1 : ***Examine the standard form of the equation***

From the equation we see that $a > 1$ so the graph is stretched vertically. We also see that $q > 0$ so the graph is shifted vertically upwards by 3 units.

Step 2 : ***Substitute values for*** θ

θ	$0°$	$30°$	$60°$	$90°$	$120°$	$150°$	$180°$	$210°$	$240°$	$270°$	$300°$	$330°$	$360°$
$f(\theta)$	5	4,73	4	3	2	1,27	1	1,27	2	3	4	4,73	5

Step 3 : ***Plot the points and join with a smooth curve***

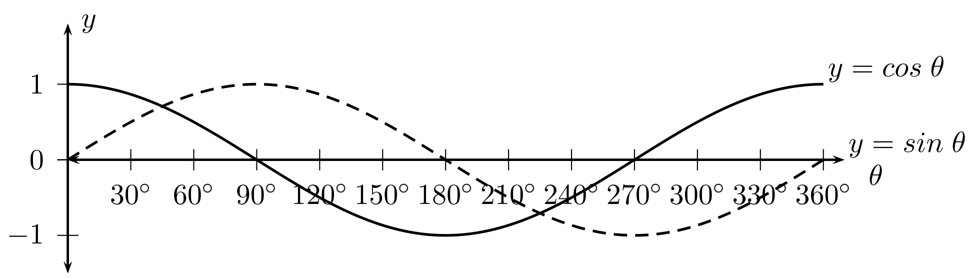


Domain: $[\ 0°; 360°]$

Range: $[1; 5]$

x-intercepts: none
y-intercept: $(0°; 5)$
Maximum turning points: $(0°; 5)$, $(360°; 5)$
Minimum turning point: $(180°; 1)$

Comparison of graphs of $y = sin\ \theta$ and $y = cos\ \theta$ EMABY



Notice that the two graphs look very similar. Both waves move up and down along the x-axis. The distances between the peaks for each graph is the same. The height of the peaks and the depths of the troughs are also the same.

If you shift the whole cosine graph to the right by $90°$ it will overlap perfectly with the sine graph. If you shift the sine graph by $90°$ to the left and it would overlap perfectly with the cosine graph. This means that:

$$\begin{aligned} sin\ \theta &= cos\ (\theta - 90°) \quad \text{(shift the cosine graph to the right)} \\ cos\ \theta &= sin\ (\theta + 90°) \quad \text{(shift the sine graph to the left)} \end{aligned}$$

Tangent function EMABZ

Functions of the form $y = tan\ \theta$ EMACA

Example 20: Plotting a tangent graph

QUESTION

$$y = f(\theta) = tan\ \theta \qquad [\,0° \leq \theta \leq 360°]$$

Use your calculator to complete the following table.
Choose an appropriate scale and plot the values with θ on the x-axis and $tan\ \theta$ on the y-axis. (Round answers to 2 decimal places).

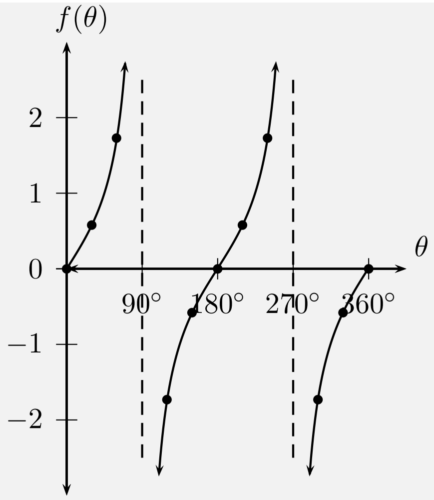
θ	$0°$	$30°$	$60°$	$90°$	$120°$	$150°$	$180°$	$210°$	$240°$	$270°$	$300°$	$330°$	$360°$
$tan\ \theta$													

SOLUTION

Step 1 : **Substitute values for θ**

θ	$0°$	$30°$	$60°$	$90°$	$120°$	$150°$	$180°$	$210°$	$240°$	$270°$	$300°$	$330°$	$360°$
$tan\ \theta$	0	$0{,}58$	$1{,}73$	undf	$-1{,}73$	$-0{,}58$	0	$0{,}58$	$1{,}73$	undf	$-1{,}73$	$-0{,}58$	0

Step 2 : **Plot the points and join with a smooth curve**



There is an easy way to visualise the tangent graph. Consider our definitions of $sin\ \theta$ and $cos\ \theta$ for right-angled triangles.

$$\frac{sin\ \theta}{cos\ \theta} = \frac{\frac{\text{opposite}}{\text{hypotenuse}}}{\frac{\text{adjacent}}{\text{hypotenuse}}} = \frac{\text{opposite}}{\text{adjacent}} = tan\ \theta$$

So for any value of θ:

$$tan\ \theta = \frac{sin\ \theta}{cos\ \theta}$$

So we know that for values of θ for which $sin\ \theta = 0$, we must also have $tan\ \theta = 0$. Also, if $cos\ \theta = 0$ the value of $tan\ \theta$ is undefined as we cannot divide by 0. The dashed vertical lines are at the values of θ where $tan\ \theta$ is not defined and are called the asymptotes.

Asymptotes: the lines $\theta = 90°$ and $\theta = 270°$
Period: $180°$
Domain: $\{\theta : 0° \leq \theta \leq 360°,\ \ \theta \neq 90°;\ 270°\}$
Range: $\{f(\theta) : f(\theta) \in \mathbb{R}\}$
x-intercepts: $(0°; 0)$, $(180°; 0)$, $(360°; 0)$
y-intercept: $(0°; 0)$

Functions of the form $y = a\ tan\ \theta + q$ EMACB

Investigation:	The effects of a and q on a tangent graph

On the same set of axes, plot the following graphs for $0° \leq \theta \leq 360°$:

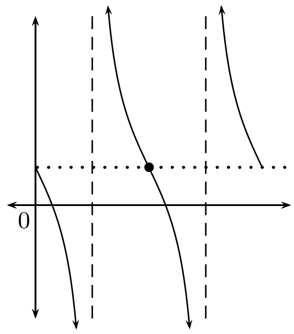
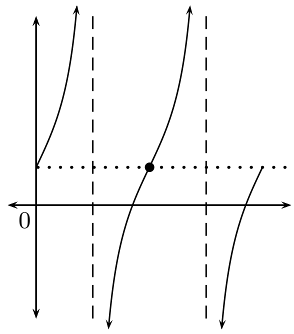
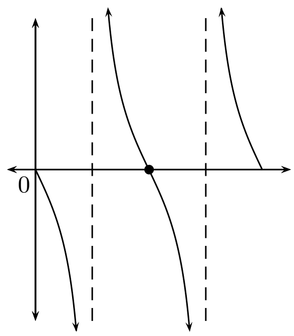
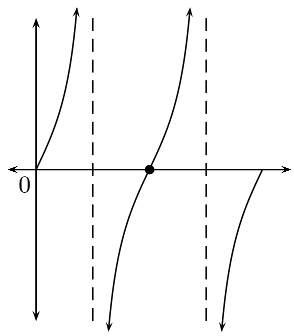
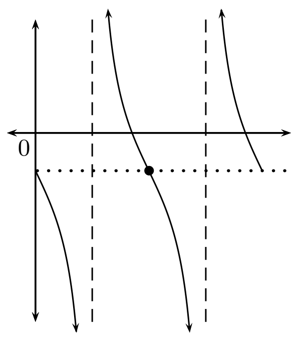
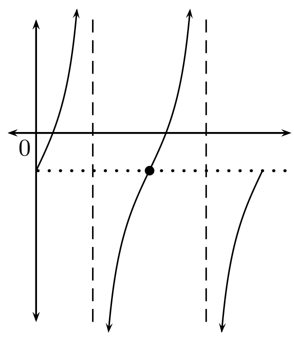
1. $y_1 = tan\ \theta - 2$
2. $y_2 = tan\ \theta - 1$
3. $y_3 = tan\ \theta$
4. $y_4 = tan\ \theta + 1$
5. $y_5 = tan\ \theta + 2$

Use your results to deduce the effect of q.

On the same set of axes, plot the following graphs for $0° \leq \theta \leq 360°$:

6. $y_6 = -2\ tan\ \theta$
7. $y_7 = -tan\ \theta$
8. $y_8 = tan\ \theta$
9. $y_9 = 2\ tan\ \theta$

Use your results to deduce the effect of a.

	$a < 0$	$a > 0$
$q > 0$		
$q = 0$		
$q < 0$		

The effect of q

The effect of q is called a vertical shift because the whole tangent graph shifts up or down by q units.

- For $q > 0$, the graph is shifted vertically upwards by q units.
- For $q < 0$, the graph is shifted vertically downwards by q units.

The effect of a

The value of a affects the steepness of each of the branches of the graph. The greater the value of a, the quicker the branches of the graph approach the asymptotes.

Discovering the characteristics EMACC

Domain and range

From the graph we see that $tan\ \theta$ is undefined at $\theta = 90°$ and $\theta = 270°$.
Therefore the domain is $\{\theta : 0° \leq \theta \leq 360°,\ \theta \neq 90°;\ 270°\}$.
The range is $\{f(\theta) : f(\theta) \in \mathbb{R}\}$.

Period

The period of $y = a\ tan\ \theta + q$ is $180°$. This means that one tangent cycle is completed in $180°$.

Intercepts

The y-intercept of $f(\theta) = a\ tan\ \theta + q$ is simply the value of $f(\theta)$ at $\theta = 0°$.

$$\begin{aligned} y &= f(0°) \\ &= a\ tan\ 0° + q \\ &= a(0) + q \\ &= q \end{aligned}$$

This gives the point $(0°; q)$.

Asymptotes

The graph has asymptotes at $\theta = 90°$ and $\theta = 270°$.

Example 21: *Sketching a tangent graph*

QUESTION

Sketch the graph of $y = 2\ tan\ \theta + 1$ *for* $\theta \in [\ 0°; 360°]$.

SOLUTION

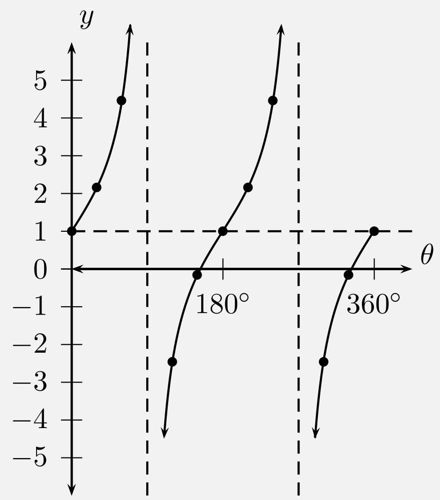
Step 1 : ***Examine the standard form of the equation***

We see that $a > 1$ so the branches of the curve will be steeper. We also see that $q > 0$ so the graph is shifted vertically upwards by 1 unit.

Step 2 : ***Substitute values for*** θ

θ	$0°$	$30°$	$60°$	$90°$	$120°$	$150°$	$180°$	$210°$	$240°$	$270°$	$300°$	$330°$	$360°$
y	1	$2{,}15$	$4{,}46$	–	$-2{,}46$	$-0{,}15$	1	$2{,}15$	$4{,}46$	–	$-2{,}46$	$-0{,}15$	1

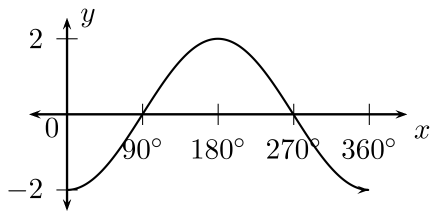
Step 3 : ***Plot the points and join with a smooth curve***

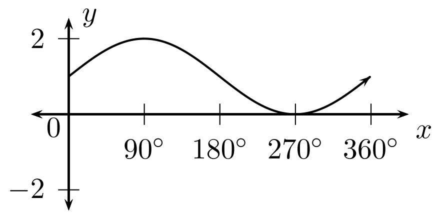


Domain: $\{\theta : 0° \leq \theta \leq 360°,\ \ \theta \neq 90°;\ 270°\}$.
Range: $\{f(\theta) : f(\theta) \in \mathbb{R}\}$.

Exercise 5 - 6

1. Using your knowledge of the effects of a and q, sketch each of the following graphs, without using a table of values, for $\theta \in [0°; 360°]$.
   (a) $y = 2\ sin\ \theta$
   (b) $y = -4\ cos\ \theta$
   (c) $y = -2\ cos\ \theta + 1$
   (d) $y = sin\ \theta - 3$
   (e) $y = tan\ \theta - 2$
   (f) $y = 2\ cos\ \theta - 1$
2. Give the equations of each of the following graphs:

   (a) 

   (b) 

More practice video solutions or help at www.everythingmaths.co.za

(1a-f.) 00fu (2a-b.) 00fv

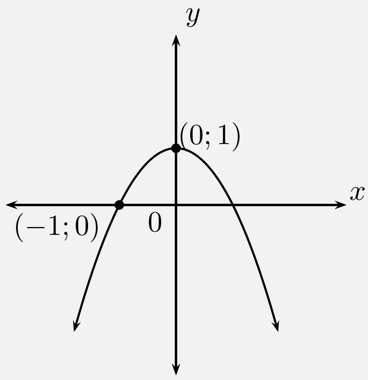
5.7 Interpretation of graphs

EMACD

Example 22: *Determining the equation of a parabola*

QUESTION

Use the sketch below to determine the values of a and q for the parabola of the form $y = ax^2 + q$.



SOLUTION

Step 1 : ***Examine the sketch***

From the sketch we see that the shape of the graph is a "frown", therefore $a < 0$. We also see that the graph has been shifted vertically upwards, therefore $q > 0$.

Step 2 : ***Determine q using the y-intercept***

The y-intercept is the point $(0; 1)$.

$$\begin{aligned} y &= ax^2 + q \\ 1 &= a(0)^2 + q \\ \therefore q &= 1 \end{aligned}$$

Step 3 : ***Use the other given point to determine a***

Substitute point $(-1; 0)$ into the equation:

$$\begin{aligned} y &= ax^2 + q \\ 0 &= a(-1)^2 + 1 \\ \therefore a &= -1 \end{aligned}$$

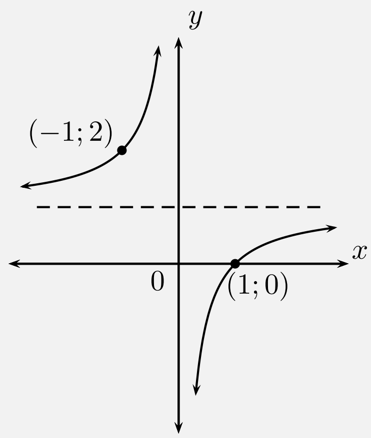
Step 4 : ***Write the final answer***

$a = -1$ and $q = 1$, so the equation of the parabola is $y = -x^2 + 1$.

Example 23: *Determining the equation of a hyperbola*

QUESTION

Use the sketch below to determine the values of a and q for the hyperbola of the form $y = \frac{a}{x} + q$.



SOLUTION

Step 1 : ***Examine the sketch***

The two curves of the hyperbola lie in the second and fourth quadrant, therefore $a < 0$. We also see that the graph has been shifted vertically upwards, therefore $q > 0$.

Step 2 : ***Substitute the given points into the equation and solve***

Substitute the point $(-1; 2)$:

$$\begin{aligned} y &= \frac{a}{x} + q \\ 2 &= \frac{a}{-1} + q \\ \therefore 2 &= -a + q \end{aligned}$$

Substitute the point $(1; 0)$:

$$\begin{aligned} y &= \frac{a}{x} + q \\ 0 &= \frac{a}{1} + q \\ \therefore a &= -q \end{aligned}$$

Step 3 : ***Solve the equations simultaneously using substitution***

$$\begin{aligned} 2 &= -a + q \\ &= q + q \\ &= 2q \\ \therefore q &= 1 \\ \therefore a &= -q \\ &= -1 \end{aligned}$$

Step 4 : ***Write the final answer***

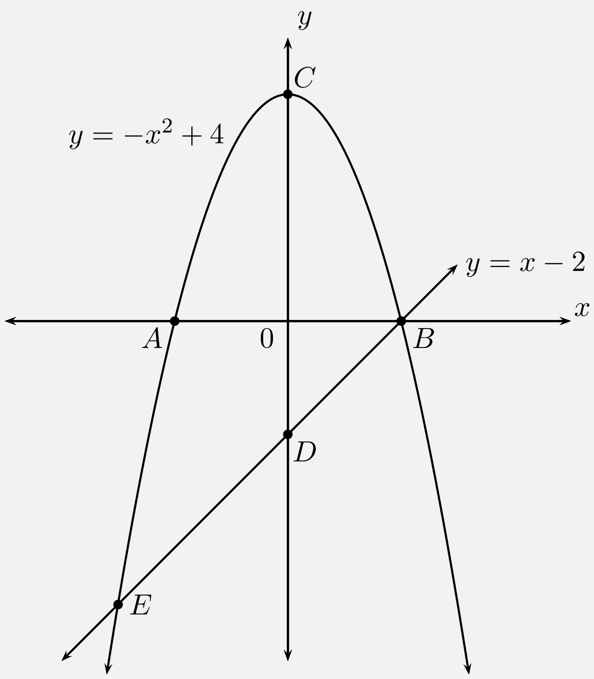
$a = -1$ and $q = 1$, the equation of the hyperbola is $y = \frac{-1}{x} + 1$.

Example 24: *Interpreting graphs*

QUESTION

The graphs of $y = -x^2 + 4$ and $y = x - 2$ are given. Calculate the following:

1. coordinates of A, B, C, D
2. coordinates of E
3. distance CD



SOLUTION

Step 1 : ***Calculate the intercepts***

For the parabola, to calculate the y-intercept, let $x = 0$:

$$\begin{aligned} y &= -x^2 + 4 \\ &= -0^2 + 4 \\ &= 4 \end{aligned}$$

This gives the point $C(0;4)$.

To calculate the x-intercept, let $y = 0$:

$$\begin{aligned} y &= -x^2 + 4 \\ 0 &= -x^2 + 4 \\ x^2 - 4 &= 0 \\ (x+2)(x-2) &= 0 \\ \therefore x &= \pm 2 \end{aligned}$$

This gives the points $A(-2;0)$ and $B(2;0)$.

For the straight line, to calculate the y-intercept, let $x = 0$:

$$\begin{aligned} y &= x - 2 \\ &= 0 - 2 \\ &= -2 \end{aligned}$$

This gives the point $D(0;-2)$. For the straight line, to calculate the x-intercept, let $y = 0$:

$$\begin{aligned} y &= x - 2 \\ 0 &= x - 2 \\ x &= 2 \end{aligned}$$

This gives the point $B(2;0)$.

Step 2 : ***Calculate the point of intersection E***

At E the two graphs intersect so we can equate the two expressions:

$$\begin{aligned} x - 2 &= -x^2 + 4 \\ \therefore x^2 + x - 6 &= 0 \\ \therefore (x-2)(x+3) &= 0 \\ \therefore x &= 2 \text{ or } -3 \end{aligned}$$

At E, $x = -3$, therefore $y = x - 2 = -3 - 2 = -5$. This gives the point $E(-3;-5)$.

Step 3 : ***Calculate distance CD***

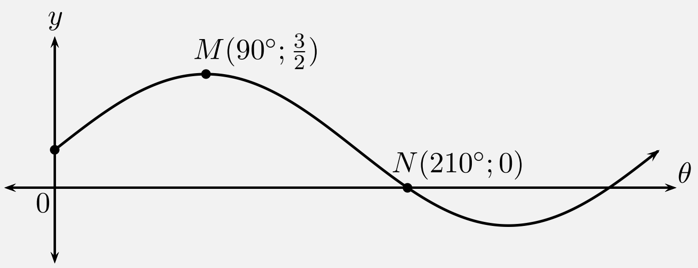
$$\begin{aligned} CD &= CO + OD \\ &= 4 + 2 \\ &= 6 \end{aligned}$$

Distance CD is 6 units.

Example 25: *Interpreting trigonometric graphs*

QUESTION

Use the sketch to determine the equation of the trigonometric function f of the form $y = a\ f(\theta) + q$.



SOLUTION

Step 1 : ***Examine the sketch***

From the sketch we see that the graph is a sine graph that has been shifted vertically upwards. The general form of the equation is $y = a\ sin\ \theta + q$.

Step 2 : ***Substitute the given points into equation and solve***

At N, $\theta = 210°$ and $y = 0$:

$$\begin{aligned} y &= a\,sin\,\theta + q \\ 0 &= a\,sin\,210° + q \\ &= a\left(-\frac{1}{2}\right) + q \\ \therefore q &= \frac{a}{2} \end{aligned}$$

At M, $\theta = 90°$ and $y = \frac{3}{2}$:

$$\begin{aligned} \frac{3}{2} &= a\,sin\,90° + q \\ &= a + q \end{aligned}$$

Step 3 : ***Solve the equations simultaneously using substitution***

$$\begin{aligned} \frac{3}{2} &= a + q \\ &= a + \frac{a}{2} \\ 3 &= 2a + a \\ 3a &= 3 \\ \therefore a &= 1 \\ \therefore q &= \frac{a}{2} \\ &= \frac{1}{2} \end{aligned}$$

Step 4 : ***Write the final answer***

$$y = sin\,\theta + \frac{1}{2}$$

Chapter 5 | Summary

Summary presentation: VMdkf at www.everythingmaths.co.za

- Characteristics of functions:
  - The given x-value is known as the independent variable, because its value can be chosen freely. The calculated y-value is known as the dependent variable, because its value depends on the x-value.
  - The domain of a function is the set of all x-values for which there exists at most one y-value according to that function. The range is the set of all y values, which can be obtained using at least one x-value.
  - An asymptote is a straight line, which the graph of a function will approach, but never touch.
  - A graph is said to be continuous if there are no breaks in the graph.
- Special functions and their properties:
  - Linear functions of the form $y = ax + q$.
  - Parabolic functions of the form $y = ax^2 + q$.
  - Hyperbolic functions of the form $y = \frac{a}{x} + q$.
  - Exponential functions of the form $y = ab^x + q$.
  - Trigonometric functions of the form
    $y = a\ sin\ \theta + q$
    $y = a\ cos\ \theta + q$
    $y = a\ tan\ \theta + q$

Chapter 5 End of Chapter Exercises

1. Sketch the graphs of the following:
   (a) $y = 2x + 4$
   (b) $y - 3x = 0$
   (c) $2y = 4 - x$
2. Sketch the following functions:
   (a) $y = x^2 + 3$
   (b) $y = \frac{1}{2}x^2 + 4$
   (c) $y = 2x^2 - 4$
3. Sketch the following functions and identify the asymptotes:
   (a) $y = 3^x + 2$
   (b) $y = -4 \times 2^x$

(c) $y = \left(\frac{1}{3}\right)^x - 2$

4. Sketch the following functions and identify the asymptotes:

   (a) $y = \frac{3}{x} + 4$

   (b) $y = \frac{1}{x}$

   (c) $y = \frac{2}{x} - 2$

5. Determine whether the following statements are true or false. If the statement is false, give reasons why:

   (a) The given or chosen y-value is known as the independent variable.

   (b) A graph is said to be congruent if there are no breaks in the graph.

   (c) Functions of the form $y = ax + q$ are straight lines.

   (d) Functions of the form $y = \frac{a}{x} + q$ are exponential functions.

   (e) An asymptote is a straight line which a graph will intersect at least once.

   (f) Given a function of the form $y = ax + q$, to find the y-intercept let $x = 0$ and solve for y.

6. Given the functions $f(x) = 2x^2 - 6$ and $g(x) = -2x + 6$:

   (a) Draw f and g on the same set of axes.

   (b) Calculate the points of intersection of f and g.

   (c) Use your graphs and the points of intersection to solve for x when:

      i. $f(x) > 0$

      ii. $g(x) < 0$

      iii. $f(x) \leq g(x)$

   (d) Give the equation of the reflection of f in the x-axis.

7. After a ball is dropped, the rebound height of each bounce decreases. The equation $y = 5(0{,}8)^x$ shows the relationship between the number of bounces x and the height of the bounce y for a certain ball. What is the approximate height of the fifth bounce of this ball to the nearest tenth of a unit?

8. Mark had 15 coins in R 5 and R 2 pieces. He had 3 more R 2 coins than R 5 coins. He wrote a system of equations to represent this situation, letting x represent the number of R 5 coins and y represent the number of R 2 coins. Then he solved the system by graphing.

   (a) Write down the system of equations.

   (b) Draw their graphs on the same set of axes.

   (c) Use your sketch to determine how many R 5 and R 2 pieces Mark had.

9. Sketch graphs of the following trigonometric functions for $\theta \in [\,0°; 360°]$. Show intercepts and asymptotes.

(a) $y = -4\ cos\ \theta$

(b) $y = sin\ \theta - 2$

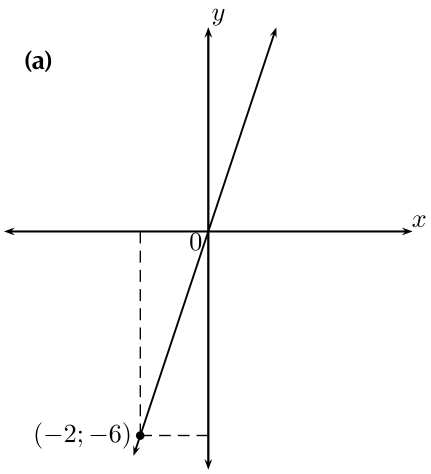
(c) $y = -2\ sin\ \theta + 1$

(d) $y = tan\ \theta + 2$

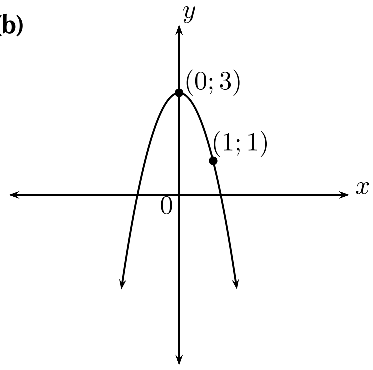
(e) $y = \dfrac{cos\ \theta}{2}$

10. Given the general equations $y = mx + c$, $y = ax^2 + q$, $y = \frac{a}{x} + q$, $y = a\ sin\ x + q$, $y = a^x + q$ and $y = a\ tan\ x$, determine the specific equations for each of the following graphs:

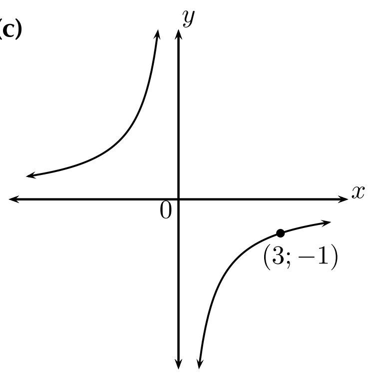
(a)



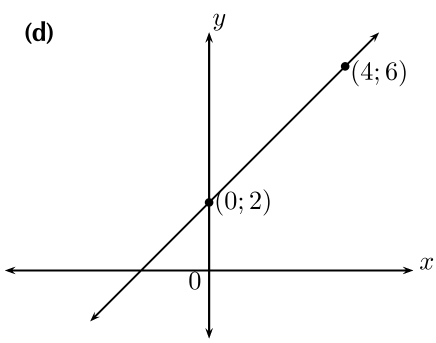
(b)



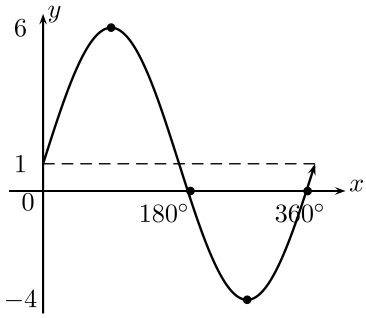
(c)



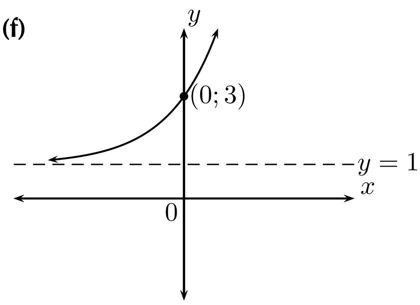
(d)



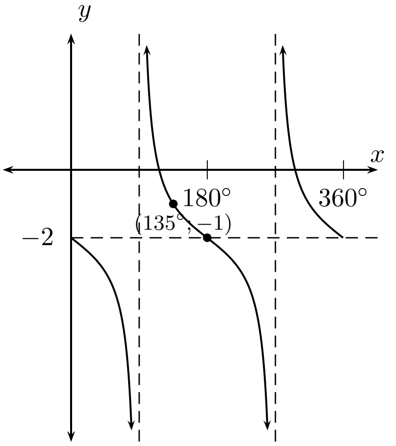
(e)



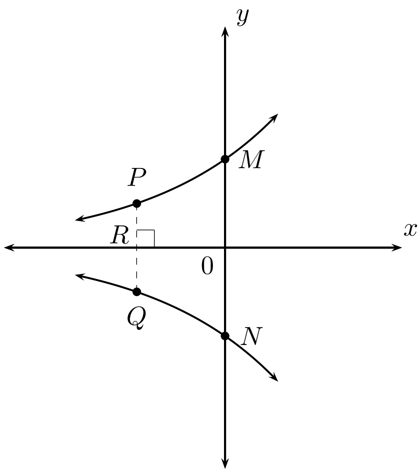
(f)



(g)



11. $y = 2^x$ and $y = -2^x$ are sketched below. Answer the questions that follow:



(a) Calculate the coordinates of M and N.

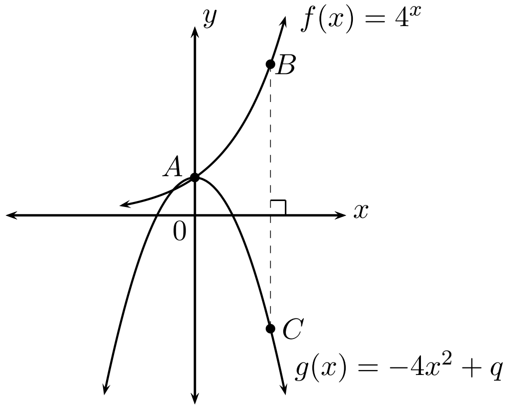
(b) Calculate the length of MN.

(c) Calculate length PQ if $OR = 1$ unit.

(d) Give the equation of $y = 2^x$ reflected about the y-axis.

(e) Give the range of both graphs.

12. $f(x) = 4^x$ and $g(x) = 4x^2 + q$ are sketched below. The points $A(0;1)$, $B(1;4)$ and C are given. Answer the questions that follow:



(a) Determine the value of q.

(b) Calculate the length of BC.

(c) Give the equation of $f(x)$ reflected about the x-axis.

(d) Give the equation of $f(x)$ shifted vertically upwards by 1 unit.

(e) Give the equation of the asymptote of $f(x)$.

(f) Give the ranges of $f(x)$ and $g(x)$.

13. Sketch the graphs $h(x) = x^2 - 4$ and $k(x) = -x^2 + 4$ on the same set of axes and answer the questions that follow:

(a) Describe the relationship between h and k.

(b) Give the equation of $k(x)$ reflected about the line $y = 4$.

(c) Give the domain and range of h.

14. Sketch the graphs $f(\theta) = 2\ sin\ \theta$ and $g(\theta) = cos\ \theta - 1$ on the same set of axes. Use your sketch to determine:

(a) $f(180°)$

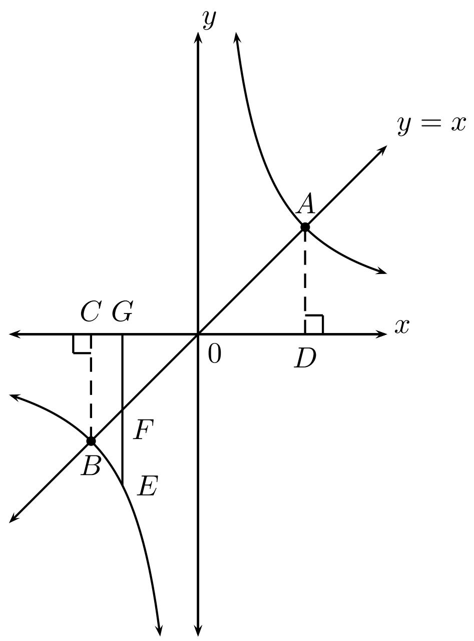
(b) $g(180°)$

(c) $g(270°) - f(270°)$

(d) The domain and range of g.

(e) The amplitude and period of f.

15. The graphs of $y = x$ and $y = \frac{8}{x}$ are shown in the following diagram.



Calculate:

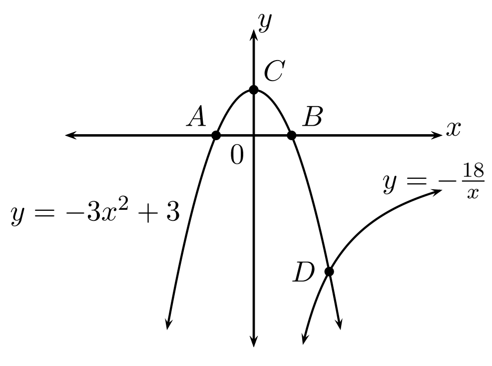
(a) the coordinates of points A and B.

(b) the length of CD.

(c) the length of AB.

(d) the length of EF, given $G(-2; 0)$.

16. Given the diagram with $y = -3x^2 + 3$ and $y = -\frac{18}{x}$.



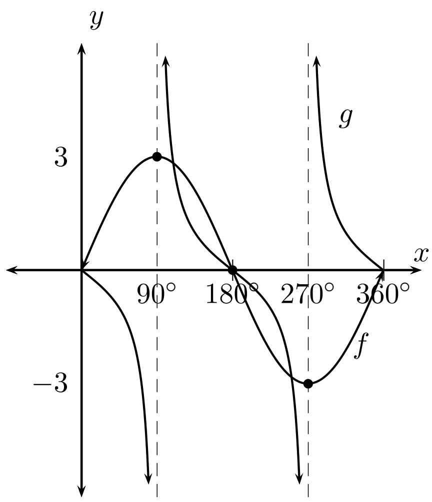
(a) Calculate the coordinates of A, B and C.

(b) Describe in words what happens at point D.

(c) Calculate the coordinates of D.

(d) Determine the equation of the straight line that would pass through points C and D .

17. The diagram shows the graphs of $f(\theta) = 3\ sin\ \theta$ and $g(\theta) = -tan\ \theta$.



(a) Give the domain of g.

(b) What is the amplitude of f?

(c) Determine for which values of θ:

i. $f(\theta) = 0 = g(\theta)$

ii. $f(\theta) \times g(\theta) < 0$

iii. $\frac{g(\theta)}{f(\theta)} > 0$

iv. $f(\theta)$ is increasing?

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(1a-c.) 00fw	(2a-c.) 00fx	(3a-c.) 00fy	(4a-c.) 00fz	(5a-f.) 00g0
(6a-d.) 00g1	(7.) 00g2	(8a-c.) 00g3	(9a-e.) 00g4	(10a-f.) 0241
(11a-e.) 00g5	(12a-f.) 00g6	(13a-c.) 00g7	(14a-e.) 00mi	(15a-d.) 02si
(16a-d.) 02sh	(17a-c.) 02sj			

Finance and growth 6

6.1 Being interested in interest EMACE

In this chapter, we apply mathematics skills to everyday financial situations.

If you had R $1\ 000$, you could either keep it in your piggy bank, or deposit it into a bank account. If you deposit the money into a bank account, you are effectively lending money to the bank and as a result, you can expect to receive interest in return. Similarly, if you borrow money from a bank, then you can expect to pay interest on the loan. Interest is charged at a percentage of the money owed over the period of time it takes to pay back the loan, meaning the longer the loan exists, the more interest will have to be paid on it.

The concept is simple, yet it is core to the world of finance. Accountants, actuaries and bankers can spend their entire working career dealing with the effects of interest on financial matters.

Video: VMbal at www.everythingmaths.co.za

6.2 Simple interest EMACF

> *DEFINITION: Simple interest*
>
> Simple interest is interest calculated only on the initial amount that you invested.

As an easy example of simple interest, consider how much we will get by investing R $1\ 000$ for 1 year with a bank that pays 5% p.a. simple interest.

At the end of the year we have

$$\begin{aligned}\text{Interest} &= \text{R } 1\ 000 \times 5\% \\ &= \text{R } 1\ 000 \times \frac{5}{100} \\ &= \text{R } 1\ 000 \times 0{,}05 \\ &= \text{R } 50\end{aligned}$$

With an opening balance of R $1\ 000$ at the start of the year, the closing balance at the end of the year will therefore be

So we can see that

$$I = P \times i$$

and

$$\begin{aligned}\text{Closing balance} &= \text{Opening balance } + \text{ Interest} \\ &= P + I \\ &= P + P \times i \\ &= P(1 + i)\end{aligned}$$

The above calculations give a good idea of what the simple interest formula looks like. However, the example shows an investment that lasts for only one year. If the investment or loan is over a longer period, we need to take this into account. We use the symbol n to indicate time period, which must be given in years.

The general formula for calculating simple interest is

$$\boxed{\begin{aligned} A &= P(1 + in) \\ \text{Where:} & \\ A &= \text{accumulated amount (final)} \\ P &= \text{principal amount (initial)} \\ i &= \text{interest written as decimal} \\ n &= \text{number of years}\end{aligned}}$$

Example 1: Calculating interest on a deposit

QUESTION

Carine deposits R 1 000 into a special bank account which pays a simple interest rate of 7% p.a. for 3 years, how much will be in her account at the end of the investment term?

SOLUTION

Step 1 : ***Write down known values***

$$\begin{aligned} P &= 1\ 000 \\ i &= 0{,}07 \\ n &= 3 \end{aligned}$$

Step 2 : ***Write down the formula***

$$A = P(1 + in)$$

Step 3 : ***Substitute the values***

$$\begin{aligned} A &= 1\ 000(1 + 0{,}07 \times 3) \\ &= 1\ 210 \end{aligned}$$

Step 4 : ***Write the final answer***

At the end of 3 years, Carine will have R 1 210 in her bank account.

Example 2: *Calculating interest on a loan*

QUESTION

Sarah borrows R $5\ 000$ from her neighbour at an agreed simple interest rate of $12{,}5\%$ p.a. She will pay back the loan in one lump sum at the end of 2 years. How much will she have to pay her neighbour?

SOLUTION

Step 1 : ***Write down the known variables***

$$\begin{aligned} P &= 5\ 000 \\ i &= 0{,}125 \\ n &= 2 \end{aligned}$$

Step 2 : ***Write down the formula***

$$A = P(1 + in)$$

Step 3 : ***Substitute the values***

$$\begin{aligned} A &= 5\ 000(1 + 0{,}125 \times 2) \\ &= 6\ 250 \end{aligned}$$

Step 4 : ***Write the final answer***

At the end of 2 years, Sarah will pay her neighbour R $6\ 250$.

We can use the simple interest formula to find pieces of missing information. For example, if we have an amount of money that we want to invest for a set amount of time to achieve a goal amount, we can rearrange the variables to solve for the required interest rate. The same principles apply to finding the length of time we would need to invest the money, if we knew the principal and accumulated amounts and the interest rate.

Important: to get a more accurate answer, try to do all your calculations on the calculator in one go. This will prevent rounding off errors from influencing your final answer.

Example 3: *Determining the investment period to achieve a goal amount*

QUESTION

Prashant deposits R $30\ 000$ into a bank account that pays a simple interest rate of $7{,}5\%$ p.a., for how many years must he invest to generate R $45\ 000$?

SOLUTION

Step 1 : ***Write down the known variables***

$$\begin{aligned} A &= 45\ 000 \\ P &= 30\ 000 \\ i &= 0{,}075 \end{aligned}$$

Step 2 : ***Write down the formula***

$$A = P(1 + in)$$

Step 3 : ***Substitute the values and solve for n***

$$
\begin{aligned}
45\ 000 &= 30\ 000(1 + 0{,}075 \times n) \\
\frac{45\ 000}{30\ 000} &= 1 + 0{,}075 \times n \\
\frac{45\ 000}{30\ 000} - 1 &= 0{,}075 \times n \\
\frac{(\frac{45\ 000}{30\ 000}) - 1}{0{,}075} &= n \\
n &= 6\frac{2}{3}
\end{aligned}
$$

Step 4 : ***Write the final answer***

It will take 6 years and 8 months to make R 45 000 from R 30 000 at a simple interest rate of 7,5% p.a.

Example 4: *Calculating the simple interest rate to achieve the desired growth*

QUESTION

At what simple interest rate should Fritha invest if she wants to grow R 2 500 to R 4 000 in 5 years?

SOLUTION

Step 1 : ***Write down the known variables***

$$
\begin{aligned}
A &= 4\ 000 \\
P &= 2\ 500 \\
n &= 5
\end{aligned}
$$

Step 2 : ***Write down the formula***

$$A = P(1 + in)$$

Step 3 : ***Substitute the values and solve for*** i

$$\begin{aligned}
4\ 000 &= 2\ 500(1 + i \times 5) \\
\frac{4\ 000}{2\ 500} &= 1 + i \times 5 \\
\frac{4\ 000}{2\ 500} - 1 &= i \times 5 \\
\frac{(\frac{4\ 000}{2\ 500}) - 1}{5} &= i \\
i &= 0{,}12
\end{aligned}$$

Step 4 : ***Write the final answer***

A simple interest rate of 12% p.a. will be needed when investing R $2\ 500$ for 5 years to become R $4\ 000$.

Exercise 6 - 1

1. An amount of R $3\ 500$ is invested in a savings account which pays simple interest at a rate of $7{,}5\%$ per annum. Calculate the balance accumulated by the end of 2 years.

2. Calculate the accumulated amount in the following situations:

   (a) A loan of R 300 at a rate of 8% for 1 year.

   (b) An investment of R $2\ 250$ at a rate of $12{,}5\%$p.a. for 6 years.

3. Sally wanted to calculate the number of years she needed to invest R $1\ 000$ for in order to accumulate R $2\ 500$. She has been offered a simple interest rate of $8{,}2\%$ p.a. How many years will it take for the money to grow to R $2\ 500$?

4. Joseph made a deposit of R $5\ 000$ in the bank for his 5 year old son's 21^{st} birthday. He has given his son the amount of R $18\ 000$ on his birthday. At what rate was the money invested, if simple interest was calculated?

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(1.) 00f5 (2.) 023y (3.) 023k (4.) 00me

6.3 Compound interest EMACG

Compound interest allows interest to be earned on interest. With simple interest, only the original investment earns interest, but with compound interest, the original investment and the interest earned on it, both earn interest. Compound interest is advantageous for investing money but not for taking out a loan.

DEFINITION: Compound interest

Compound interest is the interest earned on the principal amount and on its accumulated interest.

Consider the example of R $1\ 000$ invested for 3 years with a bank that pays 5% compound interest. At the end of the first year, the accumulated amount is

$$\begin{aligned} A_1 &= P(1+i) \\ &= 1\ 000(1+0{,}05) \\ &= 1\ 050 \end{aligned}$$

The amount A_1 becomes the new principal amount for calculating the accumulated amount at the end of the second year.

$$\begin{aligned} A_2 &= P(1+i) \\ &= 1\ 050(1+0{,}05) \\ &= 1\ 000(1+0{,}05)(1+0{,}05) \\ &= 1\ 000(1+0{,}05)^2 \end{aligned}$$

Similarly, we use the amount A_2 as the new principal amount for calculating the accumulated amount at the end of the third year.

$$\begin{aligned} A_3 &= P(1+i) \\ &= 1\ 000(1+0{,}05)^2(1+0{,}05) \\ &= 1\ 000(1+0{,}05)^3 \end{aligned}$$

Do you see a pattern? Using the formula for simple interest, we can develop a similar formula for compound interest. With an opening balance P and an interest rate of i, the closing balance at the end of the first year is:

$$\text{Closing balance after } 1 \text{ year} = P(1+i)$$

This is the same as simple interest because it only covers a single year. This closing balance becomes the opening balance for the second year of investment.

$$\begin{aligned} \text{Closing balance after } 2 \text{ years} &= [P(1+i)] \times (1+i) \\ &= P(1+i)^2 \end{aligned}$$

And similarly, for the third year

$$\begin{aligned} \text{Closing balance after } 3 \text{ years} &= [P(1+i)^2] \times (1+i) \\ &= P(1+i)^3 \end{aligned}$$

We see that the power of the term $(1+i)$ is the same as the number of years. Therefore the general formula for calculating compound interest is:

$$\boxed{\begin{aligned} A &= P(1+i)^n \\ \text{Where:} & \\ A &= \text{accumulated amount} \\ P &= \text{principal amount} \\ i &= \text{interest written as decimal} \\ n &= \text{number of years} \end{aligned}}$$

Example 5: *Compound interest*

QUESTION

Mpho wants to invest R $30\ 000$ into an account that offers a compound interest rate of 6% p.a. How much money will be in the account at the end of 4 years?

SOLUTION

Step 1 : ***Write down the known variables***

$$\begin{aligned} P &= 30\ 000 \\ i &= 0{,}06 \\ n &= 4 \end{aligned}$$

Step 2 : ***Write down the formula***

$$A = P(1+i)^n$$

Step 3 : ***Substitute the values***

$$\begin{aligned} A &= 30\ 000(1+0{,}06)^4 \\ &= 37\ 874{,}31 \end{aligned}$$

Step 4 : ***Write the final answer***

Mpho will have R $37\ 874{,}31$ in the account at the end of 4 years.

Example 6: *Calculating the compound interest rate to achieve the desired growth*

QUESTION

Charlie has been given R $5\ 000$ for his sixteenth birthday. Rather than spending it, he has decided to invest it so that he can put down a deposit of R $10\ 000$ on a car on his eighteenth birthday. What compound interest rate does he need to achieve this growth? Comment on your answer.

SOLUTION

Step 1 : ***Write down the known variables***

$$\begin{aligned} A &= 10\ 000 \\ P &= 5\ 000 \\ n &= 2 \end{aligned}$$

Step 2 : ***Write down the formula***

$$A = P(1+i)^n$$

Step 3 : ***Substitute the values and solve for i***

$$\begin{aligned} 10\ 000 &= 5\ 000(1+i)^2 \\ \frac{10\ 000}{5\ 000} &= (1+i)^2 \\ \sqrt{\frac{10\ 000}{5\ 000}} &= 1+i \\ \sqrt{\frac{10\ 000}{5\ 000}} - 1 &= i \\ i &= 0{,}4142 \end{aligned}$$

Step 4 : ***Write the final answer and comment***

Charlie needs to find an account that offers a compound interest rate of $41{,}42\%$ p.a. to achieve the desired growth. A typical savings account gives a return of approximately 2% p.a. and an aggressive investment portfolio gives a return of approximately 13% p.a. It therefore seems unlikely that Charlie will be able to invest his money at an interest rate of $41{,}42\%$ p.a.

The power of compound interest EMACH

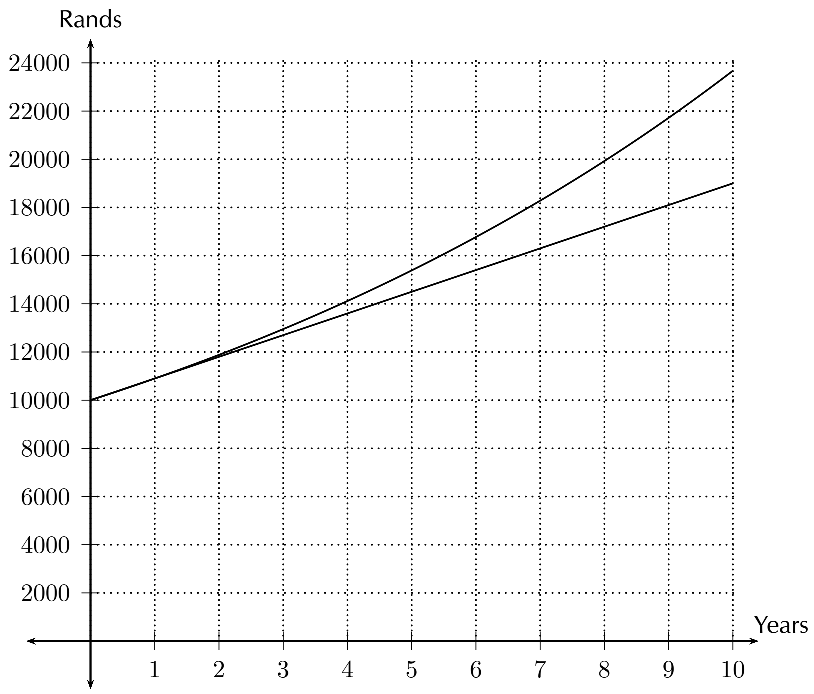
To illustrate how important "interest on interest" is, we compare the difference in closing balances for an investment earning simple interest and an investment earning compound interest. Consider an amount of R $10\ 000$ invested for 10 years, at an interest rate of 9% p.a. The closing balance for the investment earning simple interest is

$$\begin{aligned} A &= P(1 + in) \\ &= 10\ 000(1 + 0{,}09 \times 10) \\ &= \text{R } 19\ 000 \end{aligned}$$

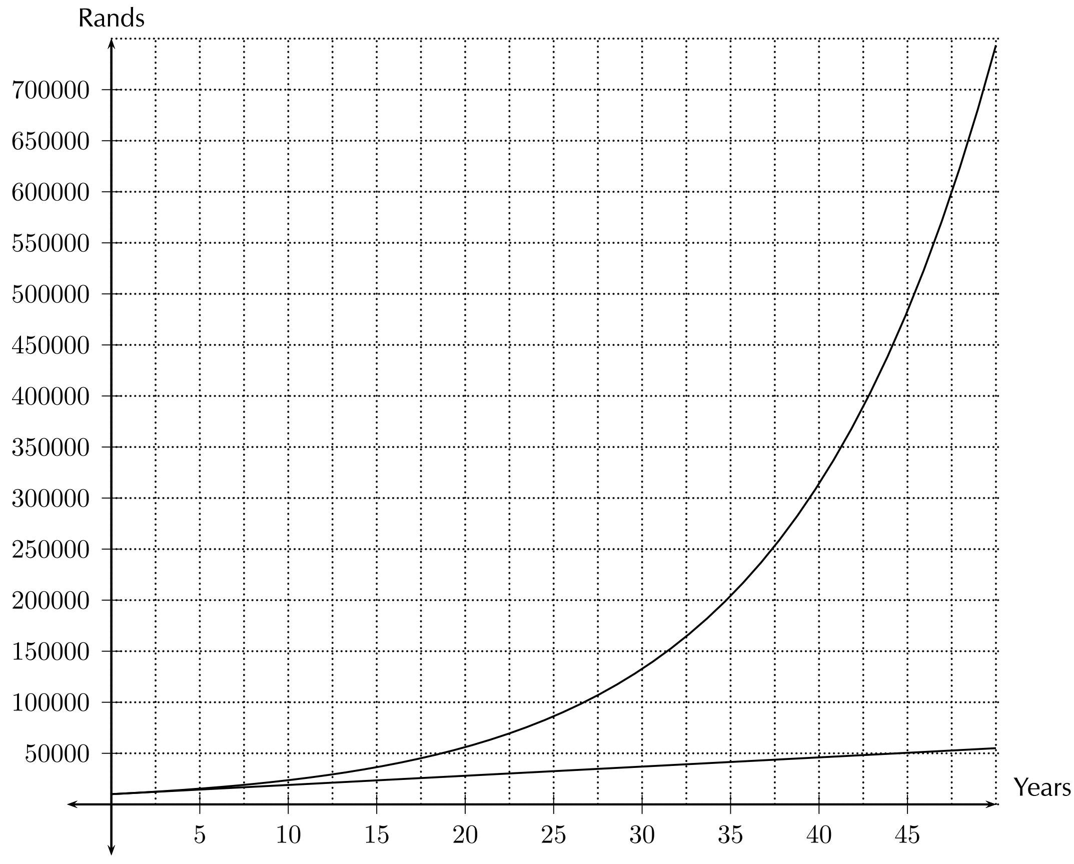
The closing balance for the investment earning compound interest is

$$\begin{aligned} A &= P(1 + i)^n \\ &= 10\ 000(1 + 0{,}09)^{10} \\ &= \text{R } 23\ 673{,}64 \end{aligned}$$

We plot the growth of the two investments on the same set of axes and note the significant different in their rate of change: simple interest is a straight line graph and compound interest is an exponential graph.



It is easier to see the vast difference in growth if we extend the time period to 50 years:



Keep in mind that this is good news and bad news. When earning interest on money invested, compound interest helps that amount to grow exponentially. But if money is borrowed the accumulated amount of money owed will increase exponentially too.

Exercise 6 - 2

1. An amount of R $3\ 500$ is invested in a savings account which pays a compound interest rate of $7{,}5\%$ p.a. Calculate the balance accumulated by the end of 2 years.
2. Morgan invests R $5\ 000$ into an account which pays out a lump sum at the end of 5 years. If he gets R $7\ 500$ at the end of the period, what compound interest rate did the bank offer him?
3. Nicola wants to invest some money at a compound interest rate of 11% p.a. How much money (to the nearest Rand) should be invested if she wants to reach a sum of R $100\ 000$ in five years time?

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(1.) 023m (2.) 00f6 (3.) 00mg

Hire purchase

EMACI

As a general rule, it is not wise to buy items on credit. When buying on credit you have to borrow money to pay for the object, meaning you will have to pay more for it due to the interest on the loan. That being said, occasionally there are appliances, such as a fridge, that are very difficult to live without. Most people don't have the cash up front to purchase such items, so they buy it on a hire purchase agreement.

A hire purchase agreement is a financial agreement between the shop and the customer about how the customer will pay for the desired product. The interest on a hire purchase loan is always charged at a simple interest rate and only charged on the amount owing. Most agreements require that a deposit is paid before the product can be taken by the customer. The principal amount of the loan is therefore the cash price minus the deposit. The accumulated loan will be worked out using the number of years the loan is needed for. The total loan amount is then divided into monthly payments over the period of the loan.

Remember: hire purchase is charged at a simple interest rate. When you are asked a hire purchase question in a test, don't forget to always use the simple interest formula.

Example 7: *Hire purchase*

QUESTION

Troy is keen to buy an additional screen for his computer, advertised for R $2\ 500$ on the Internet. There is an option of paying a 10% deposit then making 24 monthly payments using a hire purchase agreement, where interest is calculated at $7{,}5\%$ p.a. simple interest. Calculate what Troy's monthly payments will be.

SOLUTION

Step 1 : ***Write down the known variables***

A new opening balance is required, as the 10% deposit is paid in cash.

$$\begin{aligned} 10\% \text{ of } 2\ 500 &= 250 \\ \therefore P &= 2\ 500 - 250 = 2\ 250 \\ i &= 0{,}075 \\ n &= \frac{24}{12} = 2 \end{aligned}$$

Step 2 : ***Write down the formula***

$$A = P(1 + in)$$

Step 3 : ***Substitute the values***

$$
\begin{aligned}
A &= 2\ 250(1 + 0{,}075 \times 2) \\
&= 2\ 587{,}50
\end{aligned}
$$

Step 4 : ***Calculate the monthly repayments on the hire purchase agreement***

$$
\begin{aligned}
\text{Monthly payment} &= \frac{2\ 587{,}50}{24} \\
&= 107{,}81
\end{aligned}
$$

Step 5 : ***Write the final answer***

Troy's monthly payment is R $107{,}81$.

A shop can also add a monthly insurance premium to the monthly instalments. This insurance premium will be an amount of money paid monthly and gives the customer more time between a missed payment and possible repossession of the product.

Example 8: *Hire purchase with extra conditions*

QUESTION

Cassidy desperately wants to buy a TV and decides to buy one on a hire purchase agreement. The TV's cash price is R $5\ 500$. She will pay it off over 54 months at an interest rate of 21% p.a. An insurance premium of R $12{,}50$ is added to every monthly payment. How much are her monthly payments?

SOLUTION

Step 1 : ***Write down the known variables***

$$\begin{aligned} P &= 5\ 500 \\ i &= 0{,}21 \\ n &= \frac{54}{12} = 4{,}5 \end{aligned}$$

(The question does not mention a deposit, therefore we assume that Cassidy did not pay one.)

Step 2 : ***Write down the formula***

$$A = P(1 + in)$$

Step 3 : ***Substitute the values***

$$\begin{aligned} A &= 5\ 500(1 + 0{,}21 \times 4{,}5) \\ &= 10\ 697{,}50 \end{aligned}$$

Step 4 : ***Calculate the monthly repayments on the hire purchase agreement***

$$\begin{aligned} \text{Monthly payment} &= \frac{10\ 697{,}50}{54} \\ &= 198{,}10 \end{aligned}$$

Step 5 : ***Add the insurance premium***

$$198{,}10 + 12{,}50 = 210{,}60$$

Step 6 : ***Write the final answer***

Cassidy will pay R $210{,}60$ per month for 54 months until her TV is paid off.

Exercise 6 - 3

1. Vanessa wants to buy a fridge on a hire purchase agreement. The cash price of the fridge is R $4\ 500$. She is required to pay a deposit of 15% and pay the remaining loan amount off over 24 months at an interest rate of 12% p.a.
   (a) What is the principal loan amount?
   (b) What is the accumulated loan amount?
   (c) What are Vanessa's monthly repayments?
   (d) What is the total amount she has paid for the fridge?
2. Bongani buys a dining room table costing R $8\ 500$ on a hire purchase agreement. He is charged an interest rate of $17{,}5\%$ p.a. over 3 years.
   (a) How much will Bongani pay in total?
   (b) How much interest does he pay?
   (c) What is his monthly instalment?
3. A lounge suite is advertised on TV, to be paid off over 36 months at R 150 per month.
   (a) Assuming that no deposit is needed, how much will the buyer pay for the lounge suite once it has been paid off?
   (b) If the interest rate is 9% p.a., what is the cash price of the suite?

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(1.) 023z (2.) 023n (3.) 00mf

Inflation

EMACJ

There are many factors that influence the change in price of an item, one of them is inflation. Inflation is the average increase in the price of goods each year and is given as a percentage. Since the rate of inflation increases from year to year, it is calculated using the compound interest formula.

Example 9: *Calculating future cost based on inflation*

QUESTION

Milk costs R 14 for two litres. How much will it cost in 4 years time if the inflation rate is 9% p.a.?

SOLUTION

Step 1 : ***Write down the known variables***

$$\begin{aligned} P &= 14 \\ i &= 0{,}09 \\ n &= 4 \end{aligned}$$

Step 2 : ***Write down the formula***

$$A = P(1+i)^n$$

Step 3 : ***Substitute the values***

$$\begin{aligned} A &= 14(1+0{,}09)^4 \\ &= 19{,}76 \end{aligned}$$

Step 4 : ***Write the final answer***

In four years time, two litres of milk will cost R $19{,}76$.

Example 10: *Calculating past cost based on inflation*

QUESTION

A box of chocolates costs R 55 today. How much did it cost 3 years ago if the average rate of inflation was 11% p.a.?

SOLUTION

Step 1 : ***Write down the known variables***

$$\begin{aligned} A &= 55 \\ i &= 0{,}11 \\ n &= 3 \end{aligned}$$

Step 2 : ***Write down the formula***

$$A = P(1+i)^n$$

Step 3 : ***Substitute the values and solve for*** P

$$\begin{aligned} 55 &= P(1+0{,}11)^3 \\ \frac{55}{(1+0{,}11)^3} &= P \\ \therefore P &= 40{,}22 \end{aligned}$$

Step 4 : ***Write the final answer***

Three years ago, a box of chocolates would have cost R $40{,}22$.

Population growth

EMACK

Family trees increase exponentially as every person born has the ability to start another family. For this reason we calculate population growth using the compound interest formula.

Example 11: Population growth

QUESTION

If the current population of Johannesburg is $3\ 888\ 180$, and the average rate of population growth in South Africa is $2{,}1\%$ p.a., what can city planners expect the population of Johannesburg to be in 10 years?

SOLUTION

Step 1 : **Write down the known variables**

$$\begin{aligned} P &= 3\ 888\ 180 \\ i &= 0{,}021 \\ n &= 10 \end{aligned}$$

Step 2 : **Write down the formula**

$$A = P(1+i)^n$$

Step 3 : **Substitute the values**

$$\begin{aligned} A &= 3\ 888\ 180(1+0{,}021)^{10} \\ &= 4\ 786\ 343 \end{aligned}$$

Step 4 : ***Write the final answer***

City planners can expect Johannesburg's population to be $4\ 786\ 343$ in ten years time.

Exercise 6 - 4

1. If the average rate of inflation for the past few years was $7{,}3\%$ p.a. and your water and electricity account is R $1\ 425$ on average, what would you expect to pay in 6 years time?
2. The price of popcorn and a coke at the movies is now R 60. If the average rate of inflation is $9{,}2\%$ p.a. What was the price of popcorn and coke 5 years ago?
3. A small town in Ohio, USA is experiencing a huge increase in births. If the average growth rate of the population is 16% p.a., how many babies will be born to the $1\ 600$ residents in the next 2 years?

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(1.) 00f7 (2.) 00f8 (3.) 00mh

Foreign exchange rates EMACL

Different countries have their own currencies. In England, a Big Mac from McDonald's costs £ 4, in South Africa it costs R 20 and in Norway it costs 48 kr. The meal is the same in all three countries but in some places it costs more than in others. If £ $1 =$ R $12{,}41$ and 1 kr $=$ R $1{,}37$, this means that a Big Mac in England costs R $49{,}64$ and a Big Mac in Norway costs R $65{,}76$.

Exchange rates affect a lot more than just the price of a Big Mac. The price of oil increases when the South African Rand weakens. This is because when the Rand is weaker, we can buy less of other currencies with the same amount of money.

A currency gets stronger when money is invested in the country. When we buy products that are made in South Africa, we are investing in South African business and keeping the money in the country. When we buy products imported from other countries, we are investing money in those countries and as a result, the Rand will weaken. The more South African products we buy, the greater the demand for them will be and more jobs will become available for South Africans. Local is lekker!

Video: VMbcj at www.everythingmaths.co.za

Example 12: *Foreign exchange rates*

QUESTION

Saba wants to travel to see her family in Spain. She has been given R 10 000 spending money. How many Euros can she buy if the exchange rate is currently € 1 = R 10,68?

SOLUTION

Step 1 : ***Write down the equation***

Let the equivalent amount in Euros be x.

$$\begin{aligned} x &= \frac{10\ 000}{10{,}68} \\ &= 936{,}33 \end{aligned}$$

Step 2 : ***Write the final answer***

Saba can buy € 936,33 with R 10 000.

Exercise 6 - 5

1. Bridget wants to buy an iPod that costs £ 100, with the exchange rate currently at £ $1 =$ R 14. She estimates that the exchange rate will drop to R 12 in a month.
   (a) How much will the iPod cost in Rands, if she buys it now?
   (b) How much will she save if the exchange rate drops to R 12?
   (c) How much will she lose if the exchange rate moves to R 15?
2. Study the following exchange rate table:

Country	Currency	Exchange Rate
United Kingdom (UK)	Pounds (£)	R $14{,}13$
United States (USA)	Dollars ($)	R $7{,}04$

   (a) In South Africa the cost of a new Honda Civic is R $173\ 400$. In England the same vehicle costs £ $12\ 200$ and in the USA \$ $21\ 900$. In which country is the car the cheapest?
   (b) Sollie and Arinda are waiters in a South African restaurant attracting many tourists from abroad. Sollie gets a £ 6 tip from a tourist and Arinda gets \$ 12. Who got the better tip?

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(1.) 023p (2.) 0240

Chapter 6 | Summary

Summary presentation: VMdib at www.everythingmaths.co.za

- There are two types of interest rates:

Simple interest	Compound interest
$A = P(1 + in)$	$A = P(1 + i)^n$

Where:

$$\begin{aligned} A &= \text{accumulated amount} \\ P &= \text{principal amount} \\ i &= \text{interest written as decimal} \\ n &= \text{number of years} \end{aligned}$$

- Hire purchase loan repayments are calculated using the simple interest formula on the cash price, less the deposit. Monthly repayments are calculated by dividing the accumulated amount by the number of months for the repayment.
- Population growth and inflation are calculated using the compound interest formula.
- Foreign exchange rate is the price of one currency in terms of another.

Chapter 6 End of Chapter Exercises

1. Alison is going on holiday to Europe. Her hotel will cost € 200 per night. How much will she need in Rands to cover her hotel bill, if the exchange rate is € 1 = R $9{,}20$?
2. Calculate how much you will earn if you invested R 500 for 1 year at the following interest rates:
   (a) $6{,}85\%$ simple interest
   (b) $4{,}00\%$ compound interest
3. Bianca has R $1\ 450$ to invest for 3 years. Bank A offers a savings account which pays simple interest at a rate of 11% per annum, whereas Bank B offers a savings account paying compound interest at a rate of $10{,}5\%$ per annum. Which account would leave Bianca with the highest accumulated balance at the end of the 3 year period?
4. How much simple interest is payable on a loan of R $2\ 000$ for a year, if the interest rate is 10% p.a.?
5. How much compound interest is payable on a loan of R $2\ 000$ for a year, if the interest rate is 10% p.a.?
6. Discuss:
   (a) Which type of interest would you like to use if you are the borrower?
   (b) Which type of interest would you like to use if you were the banker?

7. Calculate the compound interest for the following problems.
   (a) A R $2\ 000$ loan for 2 years at 5% p.a.
   (b) A R $1\ 500$ investment for 3 years at 6% p.a.
   (c) A R 800 loan for 1 year at 16% p.a.
8. If the exchange rate to the Rand for Japanese Yen is ¥ 100 = R $6{,}2287$ and for Australian Dollar is 1 AUD = R $5{,}1094$, determine the exchange rate between the Australian Dollar and the Japanese Yen.
9. Bonnie bought a stove for R $3\ 750$. After 3 years she had finished paying for it and the R $956{,}25$ interest that was charged for hire purchase. Determine the rate of simple interest that was charged.
10. According to the latest census, South Africa currently has a population of $57\ 000\ 000$.
    (a) If the annual growth rate is expected to be $0{,}9\%$, calculate how many South Africans there will be in 10 years time (correct to the nearest hundred thousand).
    (b) If it is found after 10 years that the population has actually increased by 10 million to 67 million, what was the growth rate?

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(1.) 0235 (2.) 0236 (3.) 0237 (4.) 0238 (5.) 0239

(6.) 023a

(7.) 023b (8.) 023c (9.) 023d (10.) 023e

Trigonometry 7

Trigonometry deals with the relationship between the angles and sides of a right-angled triangle. We will learn about trigonometric functions, which form the basis of trigonometry. Video: VMbec at www.everythingmaths.co.za

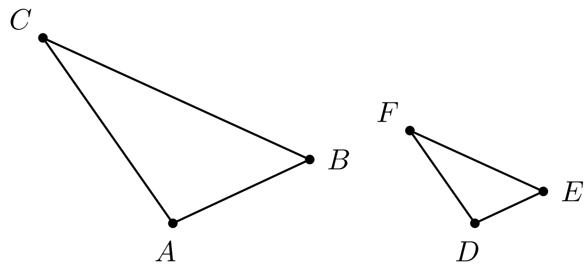
7.1 Trigonometry is useful EMACM

There are many applications of trigonometry. Of particular value is the technique of triangulation, which is used in astronomy to measure the distances to nearby stars, in geography to measure distances between landmarks, and in satellite navigation systems. GPS (the global positioning system) would not be possible without trigonometry. Other fields which make use of trigonometry include acoustics, optics, analysis of financial markets, electronics, probability theory, statistics, biology, medical imaging (CAT scans and ultrasound), chemistry, cryptology, meteorology, oceanography, land surveying, architecture, phonetics, engineering, computer graphics and game development.

7.2 Similarity of triangles EMACN

If $\triangle ABC$ is similar to $\triangle DEF$, then this is written as:

$$\triangle ABC \;|||\; \triangle DEF$$



In similar triangles, it is possible to deduce ratios between corresponding sides:

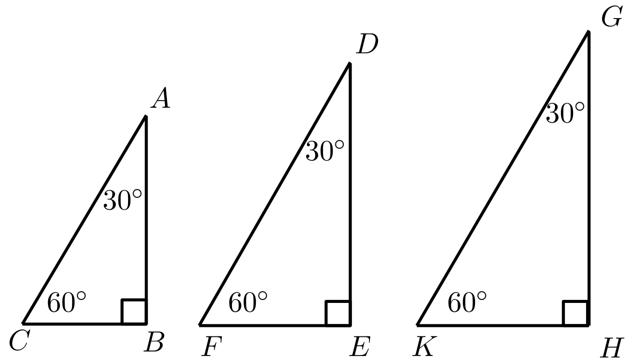
$$\frac{AB}{BC} = \frac{DE}{EF}$$
$$\frac{AB}{AC} = \frac{DE}{DF}$$
$$\frac{AC}{BC} = \frac{DF}{EF}$$
$$\frac{AB}{DE} = \frac{BC}{EF} = \frac{AC}{DF}$$

Another important fact about similar triangles ABC and DEF is that the angle at vertex A is equal to the angle at vertex D, the angle at B is equal to the angle at E, and the angle at C is equal to the angle at F.

$$\hat{A} = \hat{D}$$
$$\hat{B} = \hat{E}$$
$$\hat{C} = \hat{F}$$

Investigation:	*Ratios of similar triangles*

Draw three similar triangles of different sizes, with each triangle having interior angles equal to $30°$, $90°$ and $60°$ as shown below. Measure angles and lengths very accurately in order to fill in the table (round answers to 1 decimal place):

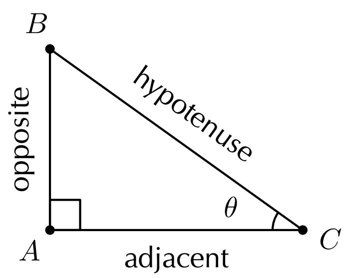


Dividing lengths of sides (ratios)		
$\frac{AB}{BC} =$	$\frac{AB}{AC} =$	$\frac{CB}{AC} =$
$\frac{DE}{EF} =$	$\frac{DE}{DF} =$	$\frac{FE}{DF} =$
$\frac{GH}{HK} =$	$\frac{GH}{GK} =$	$\frac{KH}{GK} =$

What observations can you make about the ratios of the sides?

7.3 Defining the trigonometric ratios EMACO

The ratios of similar triangles are used to define the trigonometric ratios. Consider a right-angled triangle ABC.



In the right-angled triangle, we refer to the lengths of the three sides according to how they are placed in relation to the angle θ. The side opposite to the right-angle is labelled the hypotenuse, the side opposite θ is labelled "opposite", the side next to θ is labelled "adjacent". Note that the choice of non-$90°$ internal angle is arbitrary. You can choose either internal angle and then define the adjacent and opposite sides accordingly. However, the hypotenuse remains the same regardless of which internal angle you are referring to (because it is *always* opposite the right-angle and *always* the longest side). We define the trigonometric ratios, sine, cosine and tangent of an angle, as follows:

$$sin\ \theta = \frac{\text{opposite}}{\text{hypotenuse}}$$

$$cos\ \theta = \frac{\text{adjacent}}{\text{hypotenuse}}$$

$$tan\ \theta = \frac{\text{opposite}}{\text{adjacent}}$$

These ratios, also known as trigonometric identities, relate the lengths of the sides of a right-angled triangle to its interior angles. These three ratios form the basis of trigonometry.

Important: the definitions of opposite, adjacent and hypotenuse are only applicable when working with right-angled triangles! Always check to make sure your triangle has a right-angle before you use them, otherwise you will get the wrong answer.

You may also hear people saying "Soh Cah Toa". This is a mnemonic technique for remembering the trigonometric ratios:

$\mathbf{s}\text{in}\ \theta = \frac{\mathbf{o}\text{pposite}}{\mathbf{h}\text{ypotenuse}}$
$\mathbf{c}\text{os}\ \theta = \frac{\mathbf{a}\text{djacent}}{\mathbf{h}\text{ypotenuse}}$
$\mathbf{t}\text{an}\ \theta = \frac{\mathbf{o}\text{pposite}}{\mathbf{a}\text{djacent}}$

7.4 Reciprocal functions

EMACP

Each of the three trigonometric functions has a reciprocal. The reciprocals, cosecant, secant and cotangent, are defined as follows:

$$\begin{aligned} cosec\,\theta &= \frac{1}{sin\,\theta} \\ sec\,\theta &= \frac{1}{cos\,\theta} \\ cot\,\theta &= \frac{1}{tan\,\theta} \end{aligned}$$

We can also define these reciprocals for any right-angled triangle:

$$\begin{aligned} cosec\,\theta &= \frac{\text{hypotenuse}}{\text{opposite}} \\ sec\,\theta &= \frac{\text{hypotenuse}}{\text{adjacent}} \\ cot\,\theta &= \frac{\text{adjacent}}{\text{opposite}} \end{aligned}$$

Note that:

$$\begin{aligned} sin\,\theta \times cosec\,\theta &= 1 \\ cos\,\theta \times sec\,\theta &= 1 \\ tan\,\theta \times cot\,\theta &= 1 \end{aligned}$$

Video: VMber at www.everythingmaths.co.za

Important: most scientific calculators are quite similar but these steps might differ depending on the calculator you use. Make sure your calculator is in “degrees” mode.

Example 1: *Using your calculator*

QUESTION

Use your calculator to calculate the following (correct to 2 decimal places):

1. $cos\ 48°$
2. $2\ sin\ 35°$
3. $tan^2\ 81°$
4. $3\ sin^2\ 72°$
5. $\frac{1}{4}\ cos\ 27°$
6. $\frac{5}{6}\ tan\ 34°$

SOLUTION

Step 1 :

Press $\boxed{\text{cos}}$ $\boxed{48}$ $\boxed{=}$ 0,67

Step 2 :

Press $\boxed{2}$ $\boxed{\text{sin}}$ $\boxed{35}$ $\boxed{=}$ 1,15

Step 3 :

Press $\boxed{(}$ $\boxed{\text{tan}}$ $\boxed{81}$ $\boxed{)}$ $\boxed{x^2}$ $\boxed{=}$ 39,86
OR
Press $\boxed{\text{tan}}$ $\boxed{81}$ $\boxed{=}$ $\boxed{\text{ANS}}$ $\boxed{x^2}$ $\boxed{=}$ 39,86

Step 4 :

Press $\boxed{3}$ $\boxed{(}$ $\boxed{\text{sin}}$ $\boxed{72}$ $\boxed{)}$ $\boxed{x^2}$ $\boxed{=}$ 2,71
OR
Press $\boxed{\text{sin}}$ $\boxed{72}$ $\boxed{=}$ $\boxed{\text{ANS}}$ $\boxed{x^2}$ $\boxed{=}$ $\boxed{\text{ANS}}$ $\boxed{\times}$ $\boxed{3}$

Step 5 :

Press $\boxed{(}$ $\boxed{1}$ $\boxed{\div}$ $\boxed{4}$ $\boxed{)}$ $\boxed{\text{cos}}$ $\boxed{27}$ $\boxed{=}$ 0,22
OR
Press $\boxed{\text{cos}}$ $\boxed{27}$ $\boxed{=}$ $\boxed{\text{ANS}}$ $\boxed{\div}$ $\boxed{4}$ $\boxed{=}$ 0,22

Step 6 :

Press $\boxed{(}$ $\boxed{5}$ $\boxed{\div}$ $\boxed{6}$ $\boxed{)}$ $\boxed{\text{tan}}$ $\boxed{34}$ $\boxed{=}$ $0{,}56$
OR
Press $\boxed{\text{tan}}$ $\boxed{34}$ $\boxed{=}$ $\boxed{\text{ANS}}$ $\boxed{\times}$ $\boxed{5}$ $\boxed{\div}$ $\boxed{6}$ $\boxed{=}$ $0{,}56$

Example 2: *Calculator work*

QUESTION

If $x = 25°$ and $y = 65°$, use your calculator to determine whether the following statement is true or false:

$$sin\ ^2x + cos\ ^2(90° - y) = 1$$

SOLUTION

Step 1 : ***Calculate the left hand side of the equation***

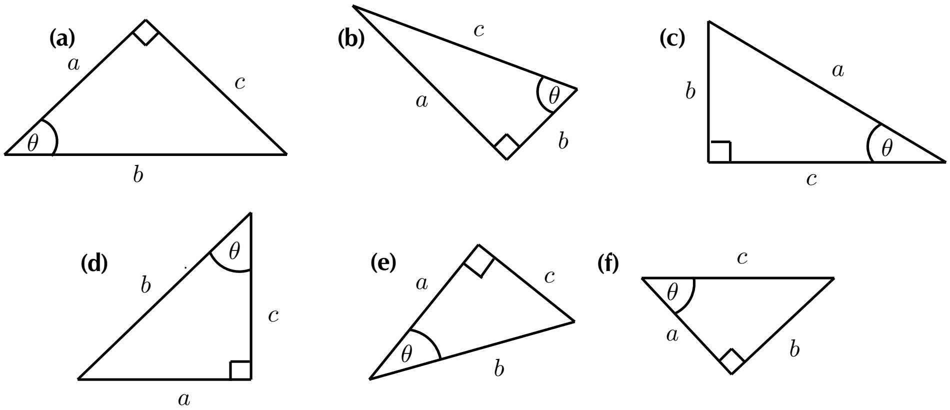
Press $\boxed{(}$ $\boxed{\text{sin}}$ $\boxed{25}$ $\boxed{)}$ $\boxed{\text{x}^2}$ $\boxed{+}$ $\boxed{(}$ $\boxed{\text{cos}}$ $\boxed{(}$ $\boxed{90}$ $\boxed{-}$ $\boxed{65}$ $\boxed{)}$ $\boxed{)}$ $\boxed{\text{x}^2}$ $\boxed{=}$
1

Step 2 : ***Write the final answer***

LHS = RHS therefore the statement is true.

Exercise 7 - 1

1. In each of the following triangles, state whether a, b and c are the hypotenuse, opposite or adjacent sides of the triangle with respect to θ.



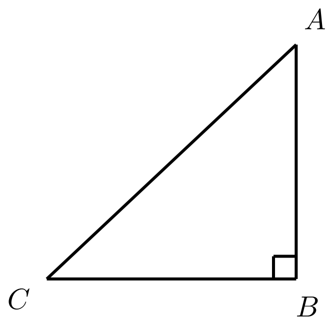
2. Use your calculator to determine the value of the following (correct to 2 decimal places):

   (a) $tan\ 65°$
   (b) $sin\ 38°$
   (c) $cos\ 74°$
   (d) $sin\ 12°$
   (e) $cos\ 26°$
   (f) $tan\ 49°$
   (g) $\frac{1}{4}\ cos\ 20°$
   (h) $3\ tan\ 40°$
   (i) $\frac{2}{3}\ sin\ 90°$

3. If $x = 39°$ and $y = 21°$, use a calculator to determine whether the following statements are true or false:

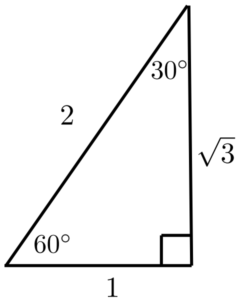
   (a) $cos\ x + 2\ cos\ x = 3\ cos\ x$
   (b) $cos\ 2y = cos\ y + cos\ y$
   (c) $tan\ x = \dfrac{sin\ x}{cos\ x}$
   (d) $cos\ (x + y) = cos\ x + cos\ y$

4. Complete each of the following (the first example has been done for you):



   (a) $sin\ \hat{A} = \frac{\text{opposite}}{\text{hypotenuse}} = \frac{CB}{AC}$
   (b) $cos\ \hat{A} =$
   (c) $tan\ \hat{A} =$
   (d) $sin\ \hat{C} =$
   (e) $cos\ \hat{C} =$
   (f) $tan\ \hat{C} =$

5. Use the triangle below to complete the following:



(a) $sin\ 60° =$

(b) $cos\ 60° =$

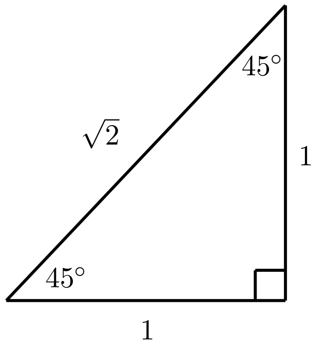
(c) $tan\ 60° =$

(d) $sin\ 30° =$

(e) $cos\ 30° =$

(f) $tan\ 30° =$

6. Use the triangle below to complete the following:



(a) $sin\ 45° =$

(b) $cos\ 45° =$

(c) $tan\ 45° =$

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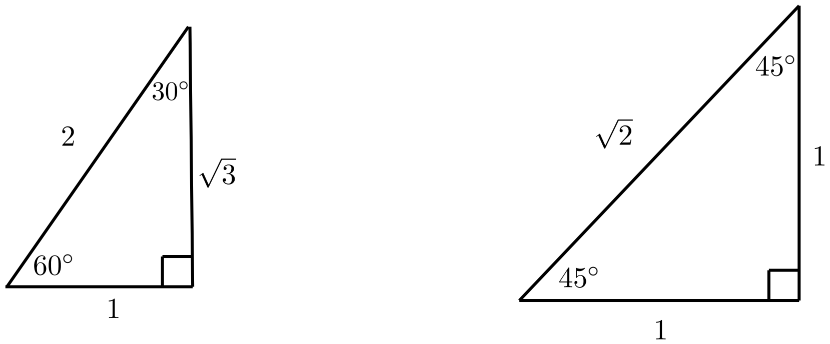
(1.) 00ke (2.) 00kf (3.) 00kg (4.) 00kh (5.) 00ki

(6.) 00kj

7.5 Special angles EMACQ

For most angles we need a calculator to calculate the values of sin , cos and tan . However, we saw in the previous exercise that we could work out these values for some special angles. The values of the trigonometric functions for these special angles are listed in the table below. Remember that the lengths of the sides of a right-angled triangle must obey the Theorem of Pythagoras: the square of the hypotenuse equals the sum of the squares of the two other sides.

θ	$30°$	$45°$	$60°$
$cos\ \theta$	$\frac{\sqrt{3}}{2}$	$\frac{1}{\sqrt{2}}$	$\frac{1}{2}$
$sin\ \theta$	$\frac{1}{2}$	$\frac{1}{\sqrt{2}}$	$\frac{\sqrt{3}}{2}$
$tan\ \theta$	$\frac{1}{\sqrt{3}}$	1	$\sqrt{3}$



These values are useful when we need to solve a problem involving trigonometric functions without using a calculator.

Exercise 7 - 2

1. Calculate the value of the following without using a calculator:
   (a) $sin\ 45° \times cos\ 45°$
   (b) $cos\ 60° + tan\ 45°$
   (c) $sin\ 60° - cos\ 60°$
2. Use the table to show that:
   (a) $\frac{sin\ 60°}{cos\ 60°} = tan\ 60°$
   (b) $sin^2 45° + cos^2 45° = 1$
   (c) $cos\ 30° = \sqrt{1 - sin^2 30°}$
3. Use the definitions of the trigonometric rations to answer the following questions:
   (a) Explain why $sin\ \alpha \leq 1$ for all values of α.
   (b) Explain why $cos\ \beta$ has a maximum value of 1.
   (c) Is there a maximum value for $tan\ \gamma$?

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(1.) 00kk (2.) 00km (3.) 00kn

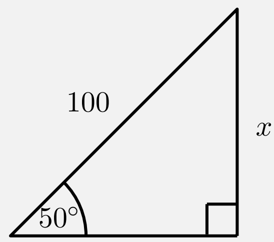
7.6 *Solving trigonometric equations* EMACR

Video: VMbfo at www.everythingmaths.co.za

Example 3: *Finding lengths*

QUESTION

Find the length of x in the following right-angled triangle:



SOLUTION

Step 1 : ***Identify the opposite and adjacent sides and the hypotenuse***

$$\sin\theta = \frac{\text{opposite}}{\text{hypotenuse}}$$

$$\sin 50^\circ = \frac{x}{100}$$

Step 2 : ***Rearrange the equation to solve for x***

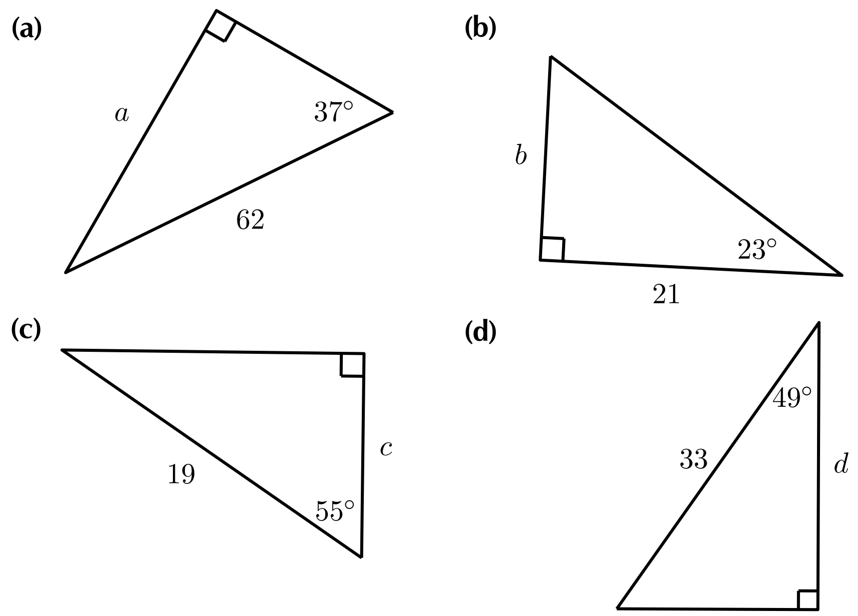
$$x = 100 \times sin\ 50°$$

Step 3 : ***Use your calculator to find the answer***

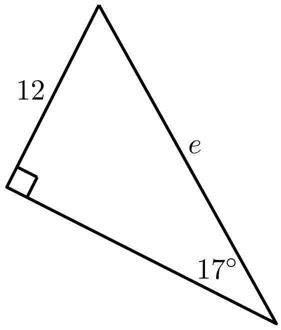
$$x = 76{,}6$$

Exercise 7 - 3

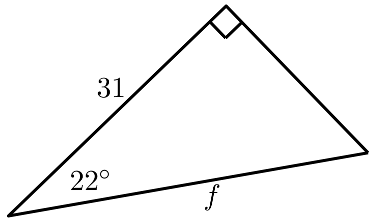
1. In each triangle find the length of the side marked with a letter. Give answers correct to 2 decimal places.



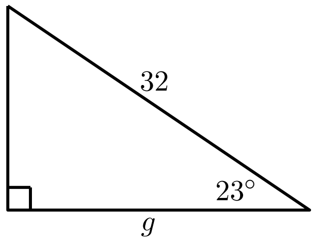
(e)



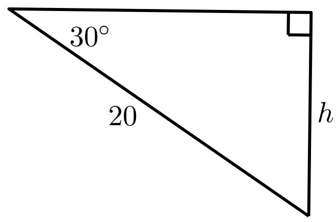
(f)



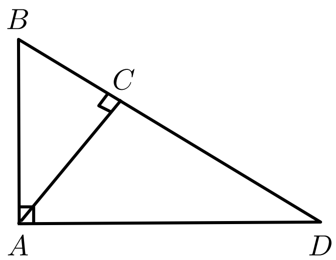
(g)



(h)



2. Write down two ratios for each of the following in terms of the sides: AB; BC; BD; AD; DC and AC:



   (a) $sin\ \hat{B}$
   (b) $cos\ \hat{D}$
   (c) $tan\ \hat{B}$

3. In $\triangle MNP$, $\hat{N} = 90°$, $MP = 20$ and $\hat{P} = 40°$. Calculate NP and MN (correct to 2 decimal places).

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(1.) 00kp (2.) 00kq (3.) 00mr

7.7 Finding an angle

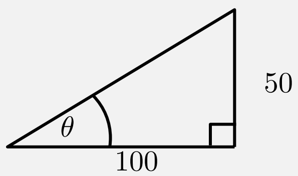
EMACS

Video: VMbfq at www.everythingmaths.co.za

Example 4: *Finding angles*

QUESTION

Find the value of θ in the following right-angled triangle:



SOLUTION

Step 1 : ***Identify the opposite and adjacent sides and the hypotenuse***

In this case you have the opposite side and the hypotenuse for angle θ.

$$\begin{aligned} tan\ \theta &= \frac{\text{opposite}}{\text{adjacent}} \\ tan\ \theta &= \frac{50}{100} \end{aligned}$$

Step 2 : ***Use your calculator to solve for θ***

To solve for θ, you will need to use the inverse function on your calculator:

Press $\boxed{\text{2ndF}}$ $\boxed{\text{tan}}$ $\boxed{(}$ $\boxed{50}$ $\boxed{\div}$ $\boxed{100}$ $\boxed{)}$ $\boxed{=}$ $26{,}6$

Step 3 : ***Write the final answer***

$$\theta = 26{,}6^{\circ}$$

Exercise 7 - 4

1. Determine the angle (correct to 1 decimal place):

(a) $tan\ \theta = 1{,}7$

(b) $sin\ \theta = 0{,}8$

(c) $cos\ \alpha = 0{,}32$

(d) $tan\ \theta = 5\frac{3}{4}$

(e) $sin\ \theta = \frac{2}{3}$

(f) $cos\ \gamma = 1{,}2$

(g) $4\ cos\ \theta = 3$

(h) $cos\ 4\theta = 0{,}3$

(i) $sin\ \beta + 2 = 2{,}65$

(j) $sin\ \theta = 0{,}8$

(k) $3\ tan\ \beta = 1$

(l) $sin\ 3\alpha = 1{,}2$

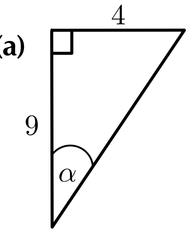
(m) $tan\ \frac{\theta}{3} = sin\ 48°$

(n) $\frac{1}{2}\ cos\ 2\beta = 0{,}3$

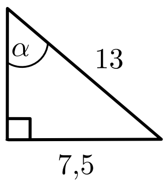
(o) $2\ sin\ 3\theta + 1 = 2{,}6$

2. Determine α in the following right-angled triangles:

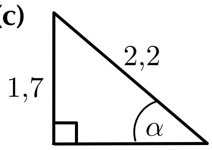
(a)



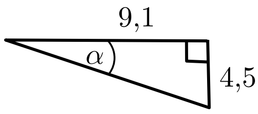
(b)



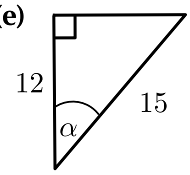
(c)



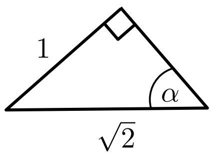
(d)



(e)



(f)



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(1.) 00kr (2.) 00ks

7.8 Two-dimensional problems EMACT

Trigonometry was developed in ancient civilisations to solve practical problems such as building construction and navigating by the stars. We will show that trigonometry can also be used to solve some other practical problems. We use the trigonometric functions to solve problems in two dimensions that involve right-angled triangles.

Video: VMbkj at www.everythingmaths.co.za

Example 5: *Flying a kite*

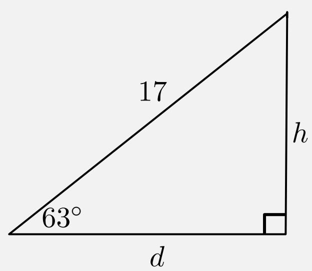
QUESTION

Mandla flies a kite on a 17 m string at an inclination of $63°$.

1. *What is the height h of the kite above the ground?*
2. *If Mandla's friend Sipho stands directly below the kite, calculate the distance d between the two friends.*

SOLUTION

Step 1 : ***Make a sketch and identify the opposite and adjacent sides and the hypotenuse***



Step 2 : ***Use given information and appropriate ratio to solve for h and d***

1.

$$\begin{aligned} sin\ 63^\circ &= \frac{\text{opposite}}{\text{hypotenuse}} \\ sin\ 63^\circ &= \frac{h}{17} \\ \therefore h &= 17\ sin\ 63^\circ \\ &= 15{,}15\text{ m} \end{aligned}$$

2.

$$\begin{aligned} cos\ 63^\circ &= \frac{\text{adjacent}}{\text{hypotenuse}} \\ cos\ 63^\circ &= \frac{d}{17} \\ \therefore d &= 17\ cos\ 63^\circ \\ &= 7{,}72\text{ m} \end{aligned}$$

Note that the third side of the triangle can also be calculated using the Theorem of Pythagoras: $d^2 = 17^2 - h^2$.

Step 3 : ***Write final answers***

1. The kite is $15{,}15$ m above the ground.
2. Mandla and Sipho are $7{,}72$ m apart.

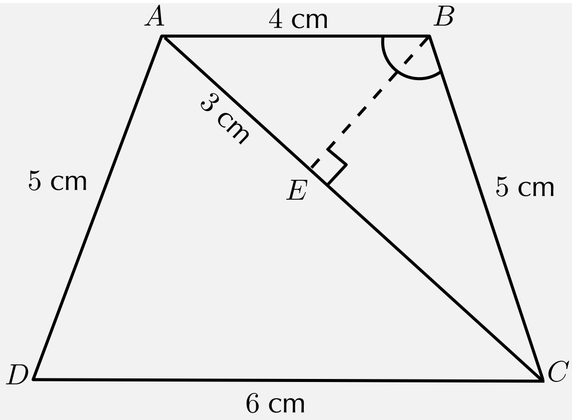
Example 6: *Calculating angles*

QUESTION

ABCD is a trapezium with $AB = 4$ cm, $CD = 6$ cm, $BC = 5$ cm and $AD = 5$ cm. Point E on diagonal AC divides the diagonal such that $AE = 3$ cm. $B\hat{E}C = 90°$. Find $A\hat{B}C$.

SOLUTION

Step 1 : ***Draw trapezium and label all given lengths on diagram. Indicate that $B\hat{E}C$ is a right-angle***



Step 2 : ***Use*** $\triangle ABE$ ***and*** $\triangle CBE$ ***to determine the two angles at*** $\hat{B}$

Step 3 : ***Find the first angle,*** $A\hat{B}E$

The hypotenuse and opposite side are given for both triangles, therefore use the sin function: In $\triangle ABE$,

$$\begin{aligned} sin\ A\hat{B}E &= \frac{\text{opposite}}{\text{hypotenuse}} \\ &= \frac{3}{4} \\ \therefore A\hat{B}E &= 48{,}6° \end{aligned}$$

Step 4 : ***Use Theorem of Pythagoras to determine*** BE

In $\triangle ABE$,

$$\begin{aligned} BE^2 &= AB^2 - AE^2 \\ &= 4^2 - 3^2 \\ &= 7 \\ \therefore BE &= \sqrt{7}\text{ cm} \end{aligned}$$

Step 5 : ***Find the second angle*** $C\hat{B}E$

In $\triangle CBE$

$$\begin{aligned} cos\ C\hat{B}E &= \frac{\text{adjacent}}{\text{hypotenuse}} \\ &= \frac{\sqrt{7}}{5} \\ &= 0{,}529 \\ \therefore C\hat{B}E &= 58{,}1° \end{aligned}$$

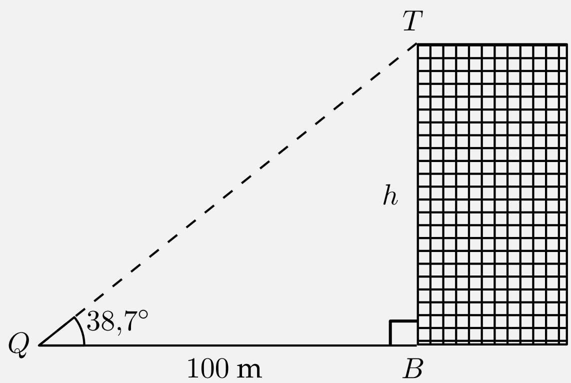
Step 6 : **Calculate the sum of the angles**

$$A\hat{B}C = 48{,}6^\circ + 58{,}1^\circ = 106{,}7^\circ$$

Another application is using trigonometry to find the height of a building. We could use a tape measure lowered from the roof, but this is impractical (and dangerous) for tall buildings. It is much more sensible to use trigonometry.

Example 7: *Finding the height of a building*

QUESTION



The given diagram shows a building of unknown height h. If we (Q) walk 100 m away from the building (B) and measure the angle from the ground to the top of the building (T), the angle is found to be $38{,}7^\circ$. This is called the angle of elevation. We have a right-angled triangle and know the length of one side and an angle. We can therefore calculate the height of the building (correct to the nearest metre).

SOLUTION

Step 1 : **Identify the opposite and adjacent sides and the hypotenuse**

Step 2 :

In $\triangle QTB$,

$$\begin{aligned} tan\ 38{,}7° &= \frac{\text{opposite}}{\text{adjacent}} \\ &= \frac{h}{100} \end{aligned}$$

Step 3 : ***Rearrange and solve for h***

$$\begin{aligned} h &= 100 \times tan\ 38{,}7° \\ &= 80{,}1 \end{aligned}$$

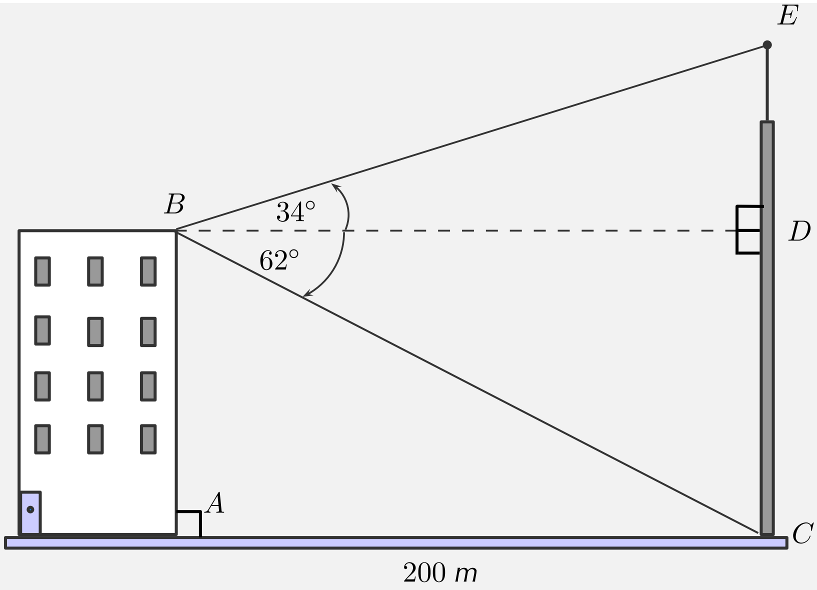
Step 4 : ***Write final answer***

The height of the building is 80 m.

Example 8: *Angles of elevation and depression*

QUESTION

A block of flats is 200 m away from a cellphone tower. Someone stands at B. They measure the angle from B to the top of the tower E to be $34°$ (the angle of elevation). They then measure the angle from B to the bottom of the tower C to be $62°$ (the angle of depression).
What is the height of the cellphone tower (correct to the nearest metre)?



Note: diagram not drawn to scale

SOLUTION

Step 1 : ***To determine height CE, first calculate lengths DE and CD***

$\triangle BDE$ and $\triangle BDC$ are both right-angled triangles. In each of the triangles, the length BD is known. Therefore we can calculate the sides of the triangles.

Step 2 : **Calculate CD**

The length AC is given. $CABD$ is a rectangle so $BD = AC = 200$ m.

In $\triangle CBD$,

$$\begin{aligned} tan\, C\hat{B}D &= \frac{CD}{BD} \\ \therefore CD &= BD \times tan\, C\hat{B}D \\ &= 200 \times tan\, 62^\circ \\ &= 376 \text{ m} \end{aligned}$$

Step 3 : **Calculate DE**

In $\triangle DBE$,

$$\begin{aligned} tan\ D\hat{B}E &= \frac{DE}{BD} \\ \therefore DE &= BD \times tan\ D\hat{B}E \\ &= 200 \times tan\ 34^\circ \\ &= 135 \text{ m} \end{aligned}$$

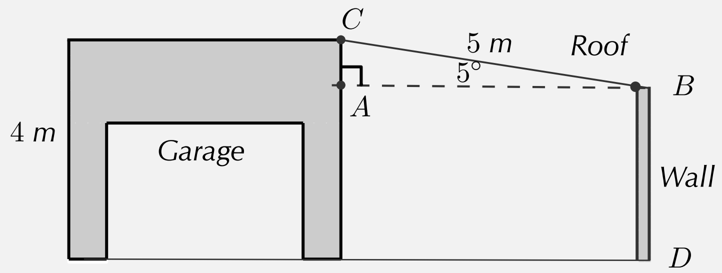
Step 4 : ***Add the two heights to get the final answer***

The height of the tower $CE = CD + DE = 135$ m$+376$ m $= 511$ m.

Example 9: *Building plan*

QUESTION

Mr Nkosi has a garage at his house and he decides to add a corrugated iron roof to the side of the garage. The garage is 4 m high, and his sheet for the roof is 5 m long. If the angle of the roof is $5°$, how high must he build the wall BD? Give the answer correct to 1 decimal place.



SOLUTION

Step 1 : ***Identify opposite and adjacent sides and hypotenuse***

$\triangle ABC$ is right-angled. The hypotenuse and an angle are known therefore we can calculate AC. The height of the wall BD is then the height of the garage minus AC.

$$\begin{aligned} sin\ A\hat{B}C &= \frac{AC}{BC} \\ \therefore AC &= BC \times sin\ A\hat{B}C \\ &= 5\ sin\ 5° \\ &= 0{,}4\text{ m} \end{aligned}$$

$$\begin{aligned} \therefore BD &= 4\text{ m} - 0{,}4\text{ m} \\ &= 3{,}6\text{ m} \end{aligned}$$

Step 2 : ***Write the final answer***

Mr Nkosi must build his wall to be $3{,}6$ m high.

Exercise 7 - 5

1. A boy flying a kite is standing 30 m from a point directly under the kite. If the kite's string is 50 m long, find the angle of elevation of the kite.
2. What is the angle of elevation of the sun when a tree $7{,}15$ m tall casts a shadow $10{,}1$ m long?
3. From a distance of 300 m, Susan looks up at the top of a lighthouse. The angle of elevation is $5°$. Determine the height of the lighthouse to the nearest metre.
4. A ladder of length 25 m is resting against a wall, the ladder makes an angle $37°$ to the wall. Find the distance between the wall and the base of the ladder.

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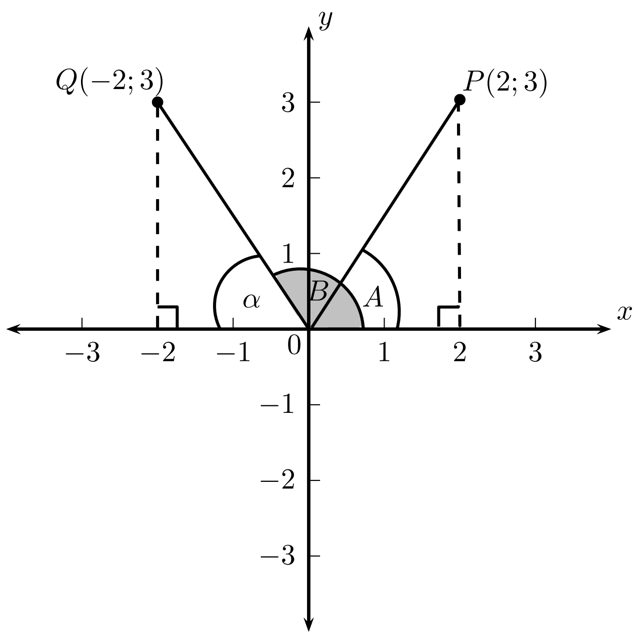
(1.) 00kt (2.) 00ku (3.) 00kv (4.) 00ms

7.9 Defining ratios in the Cartesian plane

EMACU

We have defined the trigonometric functions using right-angled triangles. We can extend these definitions to any angle, noting that the definitions do not rely on the lengths of the sides of the triangle, but on the size of the angle only. So if we plot any point on the Cartesian plane and then draw a line from the origin to that point, we can work out the angle of that line. In the figure below points P and Q have been plotted. A line from the origin (O) to each point is drawn. The dotted lines show how we can construct right-angled triangles for each point. The dotted line must always be drawn to the x-axis. Now we can find the angles A and B. We can also extend the definitions of the reciprocals in the same way:



From the coordinates of $P(2;3)$, we know the length of the side opposite $\hat{A}$ is 3 and the length of the adjacent side is 2. Using $tan\ \hat{A} = \frac{\text{opposite}}{\text{adjacent}} = \frac{3}{2}$, we calculate that $\hat{A} = 56{,}3°$.

We can also use the Theorem of Pythagoras to calculate the hypotenuse of the triangle and then calculate $\hat{A}$ using $sin\ \hat{A} = \frac{\text{opposite}}{\text{hypotenuse}}$ or $cos\ \hat{A} = \frac{\text{adjacent}}{\text{hypotenuse}}$.

Consider point $Q(-2;3)$. We define $\hat{B}$ as the angle formed between the line OQ and the positive x-axis. This is called the standard position of an angle. Angles are always measured from the positive x-axis in an anti-clockwise direction. Let $\hat{\alpha}$ be the angle formed between the line OQ and the negative x-axis such that $\hat{B} + \hat{\alpha} = 180°$.

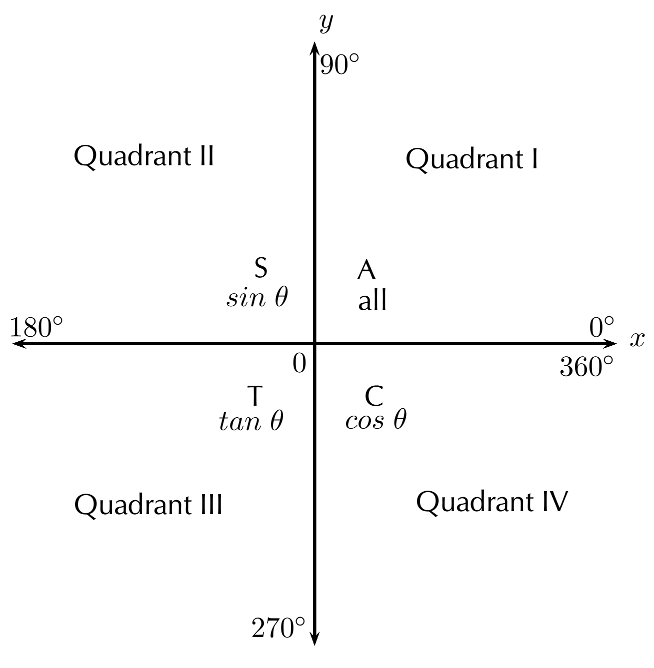
From the coordinates of $Q(-2;3)$, we know the length of the side opposite $\hat{\alpha}$ is 3 and the length of the adjacent side is 2. Using $tan\ \hat{\alpha} = \frac{\text{opposite}}{\text{adjacent}} = \frac{3}{2}$ we calculate that $\hat{\alpha} = 56{,}3°$. Therefore $\hat{B} = 180° - \hat{\alpha} = 123{,}7°$.

Similarly, an alternative method is to calculate the hypotenuse using the Theorem of Pythagoras and calculate $\hat{\alpha}$ using $sin\ \hat{\alpha} = \frac{\text{opposite}}{\text{hypotenuse}}$ or $cos\ \hat{\alpha} = \frac{\text{adjacent}}{\text{hypotenuse}}$.

If we were to draw a circle centred on the origin (O) and passing through point P, then the length from the origin to point P is the radius of the circle, which we denote r. We can rewrite all the trigonometric functions in terms of x, y and r. The general definitions for the trigonometric functions are:

$$\begin{array}{rclrcl} sin\ \theta & = & \frac{y}{r} & cosec\ \theta & = & \frac{r}{y} \\ cos\ \theta & = & \frac{x}{r} & sec\ \theta & = & \frac{r}{x} \\ tan\ \theta & = & \frac{y}{x} & cot\ \theta & = & \frac{x}{y} \end{array}$$

The Cartesian plane is divided into 4 quadrants in an anti-clockwise direction as shown in the diagram below. Notice that r is always positive but the values of x and y change depending on the position of the point in the Cartesian plane. As a result, the trigonometric ratios can be positive or negative. The letters C, A, S and T indicate which of the functions are positive in each quadrant:



This diagram is known as the CAST diagram.

Quadrant I: All ratios are positive.
Quadrant II: y values are positive therefore sine and cosine are positive.
Quadrant III: Both x and y values are negative therefore tangent and cotangent are positive.
Quadrant IV: x values are positive therefore cosecant and secant are positive.

Note: the hypotenuse r is a length, and is therefore always positive. Video: VMbhy at www.everythingmaths.co.za

Special angles in the Cartesian plane

When working in the Cartesian plane we include two other special angles in right-angled triangles: $0°$ and $90°$. These are special since we can usually not have an angle of $0°$ or another angle (in addition to the right angle) of $90°$. Notice that when $\theta = 0°$, the length of the opposite side is equal to 0, therefore

$$sin\ 0° = \frac{\text{opposite}}{\text{hypotenuse}} = \frac{0}{\text{hypotenuse}} = 0.$$

When $\theta = 90°$, the length of the adjacent side is equal to 0, therefore

$$cos\ 90° = \frac{\text{adjacent}}{\text{hypotenuse}} = \frac{0}{\text{hypotenuse}} = 0.$$

Using the definition $tan\ \theta = \frac{sin\ \theta}{cos\ \theta}$ we see that for $\theta = 0°$, $sin\ 0° = 0$, therefore

$$tan\ 0° = \frac{0}{cos\ 0°} = 0.$$

For $\theta = 90°$, $cos\ 90° = 0$, therefore

$$tan\ 90° = \frac{sin\ 90°}{0} = \text{undefined}.$$

θ	$0°$	$30°$	$45°$	$60°$	$90°$
$cos\ \theta$	1	$\frac{\sqrt{3}}{2}$	$\frac{1}{\sqrt{2}}$	$\frac{1}{2}$	0
$sin\ \theta$	0	$\frac{1}{2}$	$\frac{1}{\sqrt{2}}$	$\frac{\sqrt{3}}{2}$	1
$tan\ \theta$	0	$\frac{1}{\sqrt{3}}$	1	$\sqrt{3}$	undef

Example 10: *Ratios in the Cartesian plane*

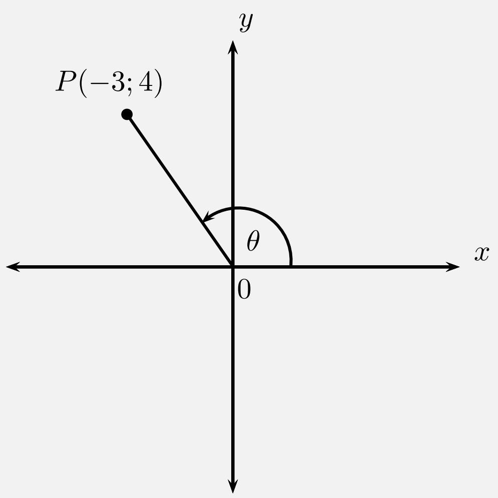
QUESTION

$P(-3;4)$ *is a point in the Cartesian plane.* $X\hat{O}P = \theta$, *where* X *is a point on the* x*-axis. Without using a calculator, determine the value of:*

1. $cos\ \theta$
2. $3\ tan\ \theta$
3. $\frac{1}{2}\ cosec\ \theta$

SOLUTION

Step 1 : **Sketch point P in the Cartesian plane and label θ**



Step 2 : **Use Pythagoras to calculate r**

$$\begin{aligned} r^2 &= x^2 + y^2 \\ &= (-3)^2 + (4)^2 \\ &= 25 \\ \therefore r &= 5 \end{aligned}$$

Step 3 : **Substitute values for x, y and r into the required ratios**

1. $cos\ \theta = \frac{x}{r} = -\frac{3}{5}$

2. $3\ tan\ \theta = 3\left(\frac{y}{x}\right) = 3\left(\frac{4}{-3}\right) = -4$

3. $\frac{1}{2}\ cosec\ \theta = \frac{1}{2}\left(\frac{r}{y}\right) = \frac{1}{2}\left(\frac{5}{4}\right) = \frac{5}{8}$

Example 11: Ratios in the Cartesian plane

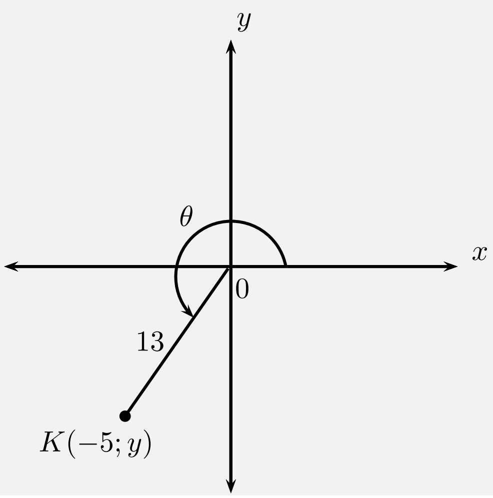
QUESTION

$X\hat{O}K = \theta$ is an angle in the third quadrant and K is the point $(-5; y)$. OK is 13 units. Determine without using a calculator:

1. *The value of y*
2. *Prove that $tan^{\ 2}\ \theta + 1 = sec^{\ 2}\ \theta$*

SOLUTION

Step 1 : ***Sketch point K in the Cartesian plane and label θ***



Step 2 : ***Use Pythagoras to calculate y***

$$\begin{aligned} r^2 &= x^2 + y^2 \\ y^2 &= r^2 - x^2 \\ &= (13)^2 - (-5)^2 \\ &= 169 - 25 \\ &= 144 \\ y &= \pm 12 \end{aligned}$$

Given that θ lies in the third quadrant, y must be negative.

$$\therefore y = -12$$

Step 3 : ***Substitute values for x, y and r and simplify***

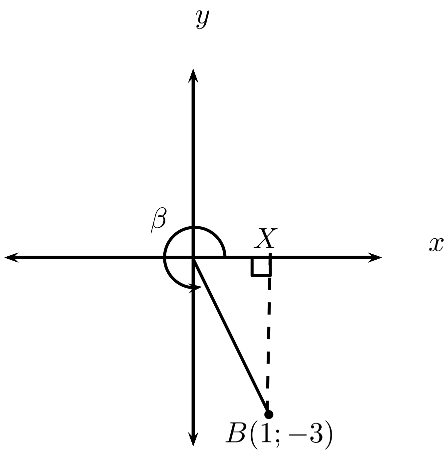
$$\begin{aligned} \text{LHS} &= tan^{\,2}\,\theta + 1 & \text{RHS} &= sec^{\,2}\,\theta \\ &= \left(\frac{y}{x}\right)^2 + 1 & &= \left(\frac{r}{x}\right)^2 \\ &= \left(\frac{-12}{-5}\right)^2 + 1 & &= \left(\frac{13}{-5}\right)^2 \\ &= \left(\frac{144}{25}\right) + 1 & &= \frac{169}{25} \end{aligned}$$

$$\begin{aligned} &= \frac{144 + 25}{25} \\ &= \frac{169}{25} \\ \therefore \text{LHS} &= \text{RHS} \end{aligned}$$

Important: whenever you have to solve trigonometric problems without a calculator, always make a sketch.

Exercise 7 - 6

1. B is a point in the Cartesian plane. Determine without using a calculator:
   (a) OB
   (b) $cos\ \beta$
   (c) $cosec\ \beta$
   (d) $tan\ \beta$



2. If $sin\ \theta = 0{,}4$ and θ is an obtuse angle, determine:
   (a) $cos\ \theta$
   (b) $\sqrt{21}\ tan\ \theta$
3. Given $tan\ \theta = \frac{t}{2}$, where $0° \leq \theta \leq 90°$. Determine the following in terms of t:
   (a) $sec\ \theta$
   (b) $cot\ \theta$
   (c) $cos^2\ \theta$
   (d) $tan^2\ \theta - sec^2\ \theta$

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Chapter 7 | Summary

Summary presentation: VMdwh at www.everythingmaths.co.za

- We can define three trigonometric ratios for right-angled triangles: sine (sin), cosine (cos) and tangent (tan).
- Each of these ratios have a reciprocal: cosecant ($cosec$), secant (sec) and cotangent (cot).
- We can use the principles of solving equations and the trigonometric ratios to help us solve simple trigonometric equations.
- We can solve problems in two dimensions that involve right-angled triangles.
- For some special angles, we can easily find the values of sin, cos and tan without using a calculator.
- We can extend the definitions of the trigonometric functions to any angle.
- Trigonometry is used to help us solve problems in two dimensions, such as finding the height of a building.

Chapter 7 End of Chapter Exercises

1. Without using a calculator determine the value of

$$sin\ 60° cos\ 30° - cos\ 60° sin\ 30° + tan\ 45°$$

2. If $3\ tan\ \alpha = -5$ and $0° < \alpha < 270°$, use a sketch to determine:

   (a) $cos\ \alpha$

   (b) $tan^{\ 2}\ \alpha - sec^{\ 2}\ \alpha$

3. Solve for θ if θ is a positive, acute angle:

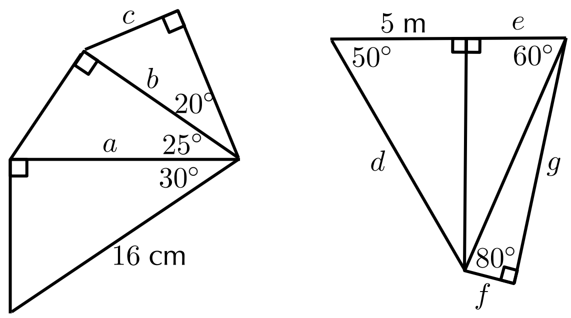
   (a) $2\ sin\ \theta = 1{,}34$

   (b) $1 - tan\ \theta = -1$

   (c) $cos\ 2\theta = sin\ 40°$

   (d) $\dfrac{sin\ \theta}{cos\ \theta} = 1$

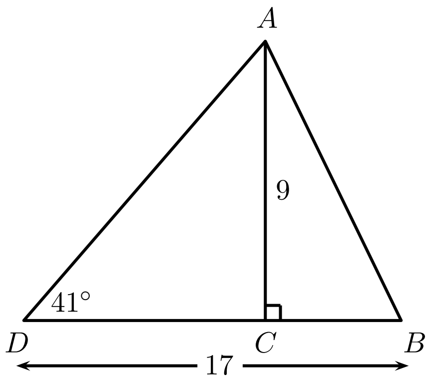
4. Calculate the unknown lengths in the diagrams below:



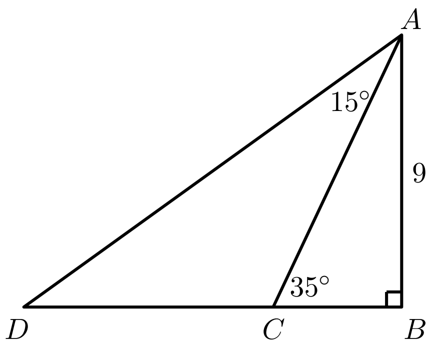
5. In $\triangle PQR$, $PR = 20$ cm, $QR = 22$ cm and $P\hat{R}Q = 30°$. The perpendicular line from P to QR intersects QR at X. Calculate:
   (a) the length XR
   (b) the length PX
   (c) the angle $Q\hat{P}X$

6. A ladder of length 15 m is resting against a wall, the base of the ladder is 5 m from the wall. Find the angle between the wall and the ladder.

7. In the following triangle find the angle $A\hat{B}C$:



8. In the following triangle find the length of side CD:



9. Given $A(5; 0)$ and $B(11; 4)$, find the angle between the line through A and B and the x-axis.

10. Given $C(0; -13)$ and $D(-12; 14)$, find the angle between the line through C and D and the y-axis.

11. A right-angled triangle has hypotenuse 13 mm. Find the length of the other two sides if one of the angles of the triangle is $50°$.

12. One of the angles of a rhombus with perimeter 20 cm is $30°$.

(a) Find the sides of the rhombus.

(b) Find the length of both diagonals.

13. Captain Jack is sailing towards a cliff with a height of 10 m.

(a) The distance from the boat to the top of the cliff is 30 m. Calculate the angle of elevation from the boat to the top of the cliff (correct to the nearest integer).

(b) If the boat sails 7 m closer to the cliff, what is the new angle of elevation from the boat to the top of the cliff?

14. Given the points, $E(5;0)$, $F(6;2)$ and $G(8;-2)$, find the angle $F\hat{E}G$.

15. A triangle with angles $40°$, $40°$ and $100°$ has a perimeter of 20 cm. Find the length of each side of the triangle.

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(1.) 00ky	(2.) 00kz	(3.) 00m0	(4.) 00m1	(5.) 00m2
(6.) 00m3	(7.) 00m4	(8.) 00m5	(9.) 00m6	(10.) 00m7
(11.) 00m8	(12.) 00m9	(13.) 00ma	(14.) 00mb	(15.) 00mc

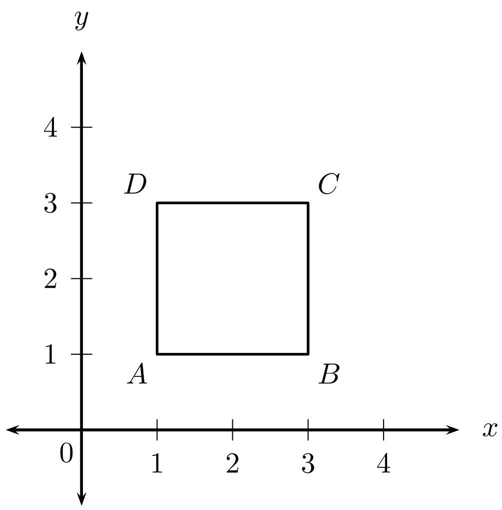
Analytical geometry 8

Analytical geometry is the study of geometric properties, relationships and measurement of points, lines and angles in the Cartesian plane. Geometrical shapes are defined using a coordinate system and algebraic principles. Some consider the introduction of analytical geometry, also called coordinate or Cartesian geometry, to be the beginning of modern mathematics.

Video: VMbkm at www.everythingmaths.co.za

8.1 Drawing figures on the Cartesian plane EMACV

If we are given the coordinates of the vertices of a figure, we can draw the figure on the Cartesian plane. For example, quadrilateral $ABCD$ with coordinates $A(1;1)$, $B(3;1)$, $C(3;3)$ and $D(1;3)$.



The order of the letters for naming a figure is important. It indicates the order in which points must be joined: A to B, B to C, C to D and D back to A. It would also be correct to write quadrilateral $CBAD$ or $BADC$ but it is better to follow the convention of writing letters in alphabetical order.

8.2 Distance between two points

EMACW

A point is a simple geometric object having location as its only property.

DEFINITION: Point

A point is an ordered pair of numbers written as $(x; y)$.

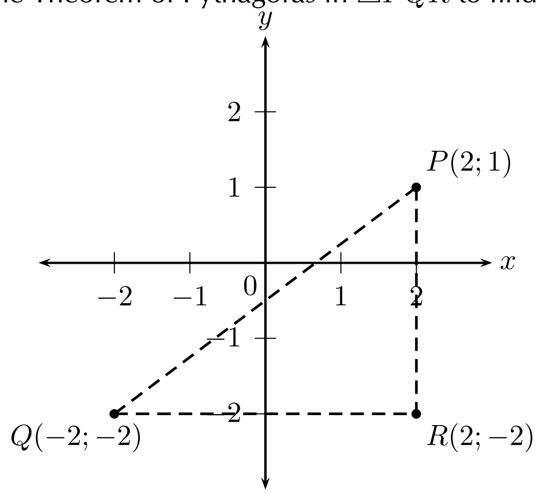
DEFINITION: Distance

Distance is a measure of the length between two points.

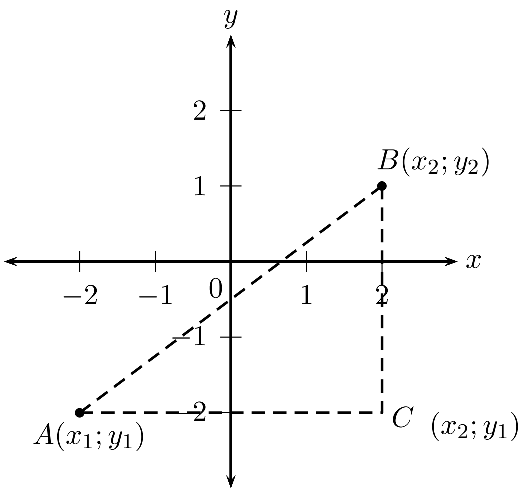
Investigation:	*Distance between two points*

Points $P(2; 1)$, $Q(-2; -2)$ and $R(2, -2)$ are given.

- Can we assume that $\hat{R} = 90°$? If so, why?
- Apply the Theorem of Pythagoras in $\triangle PQR$ to find the length of PQ.



To derive a general formula for the distance between two points $A(x_1; y_1)$ and $B(x_2; y_2)$, we use the Theorem of Pythagoras:



$$AB^2 = AC^2 + BC^2$$
$$\therefore AB = \sqrt{AC^2 + BC^2}$$

And

$$\begin{aligned} AC &= x_2 - x_1 \\ BC &= y_2 - y_1 \end{aligned}$$

Therefore,

$$\begin{aligned} AB &= \sqrt{AC^2 + BC^2} \\ &= \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2} \end{aligned}$$

Therefore to calculate the distance between any two points, $(x_1; y_1)$ and $(x_2; y_2)$, we use:

$$\text{Distance } (d) = \sqrt{(x_1 - x_2)^2 + (y_1 - y_2)^2}$$

Notice that $(x_1 - x_2)^2 = (x_2 - x_1)^2$.

Video: VMbls at www.everythingmaths.co.za

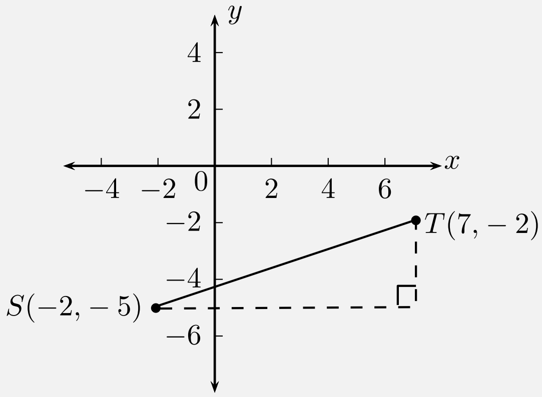
Example 1: *Using the distance formula*

QUESTION

Find the distance between $S(-2;-5)$ and $Q(7;-2)$.

SOLUTION

Step 1 : ***Draw a sketch***



Step 2 : ***Assign values to** $(x_1;y_1)$ **and** $(x_2;y_2)$*

Let the coordinates of S be $(x_1;y_1)$ and the coordinates of T be $(x_2;y_2)$.

$$x_1 = -2 \qquad y_1 = -5 \qquad x_2 = 7 \qquad y_2 = -2$$

Step 3 : ***Write down the distance formula***

$$d = \sqrt{(x_1 - x_2)^2 + (y_1 - y_2)^2}$$

Step 4 : ***Substitute values***

$$
\begin{aligned}
d_{ST} &= \sqrt{(-2-7)^2+(-5-(-2))^2} \\
&= \sqrt{(-9)^2+(-3)^2} \\
&= \sqrt{81+9} \\
&= \sqrt{90} \\
&= 9{,}5
\end{aligned}
$$

Step 5 : ***Write the final answer***

The distance between S and T is $9{,}5$ units.

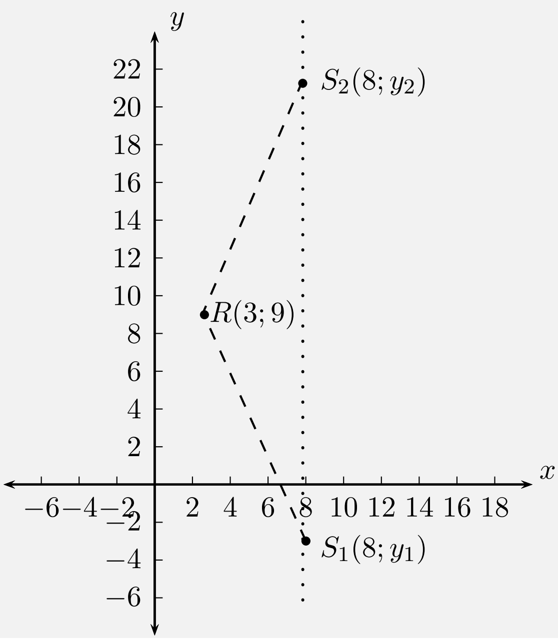
Example 2: *Using the distance formula*

QUESTION

Given $RS = 13$, $R(3;9)$ and $S(8;y)$, find y.

SOLUTION

Step 1 : ***Draw a sketch***



Step 2 : ***Assign values to*** $(x_1; y_1)$ ***and*** $(x_2; y_2)$

Let the coordinates of R be $(x_1; y_1)$ and the coordinates of S be $(x_2; y_2)$.

$$x_1 = 3 \qquad y_1 = 9 \qquad x_2 = 8 \qquad y_2 = y$$

Step 3 : ***Write down the distance formula***

$$d = \sqrt{(x_1 - x_2)^2 + (y_1 - y_2)^2}$$

Step 4 : ***Substitute values and solve for*** y

$$\begin{aligned}
13 &= \sqrt{(3-8)^2 + (9-y)^2} \\
13^2 &= (-5)^2 + (81 - 18y + y^2) \\
0 &= y^2 - 18y - 63 \\
&= (y+3)(y-21) \\
\therefore y &= -3 \text{ or } y = 21
\end{aligned}$$

Step 5 : ***Write the final answer***

S is $(8; -3)$ or $(8; 21)$

Important: always draw a sketch. It helps with your calculation and makes it easier to check if your answer is correct.

Exercise 8 - 1

1. Find the length of AB if:
   - **(a)** $A(2; 7)$ and $B(-3; 5)$
   - **(b)** $A(-3; 5)$ and $B(-9; 1)$
   - **(c)** $A(x; y)$ and $B(x + 4; y - 1)$
2. The length of $CD = 5$. Find the missing coordinate if:
   - **(a)** $C(6; -2)$ and $D(x; 2)$

(b) $C(4; y)$ and $D(1; -1)$

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(1.) 00cy (2.) 00cz

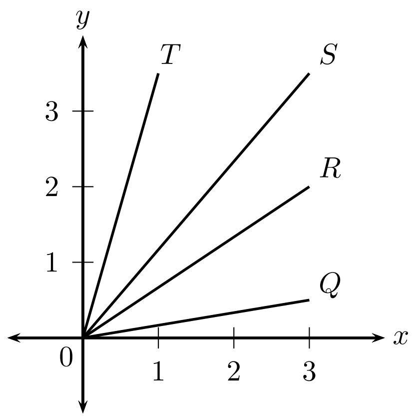
8.3 Gradient of a line

EMACX

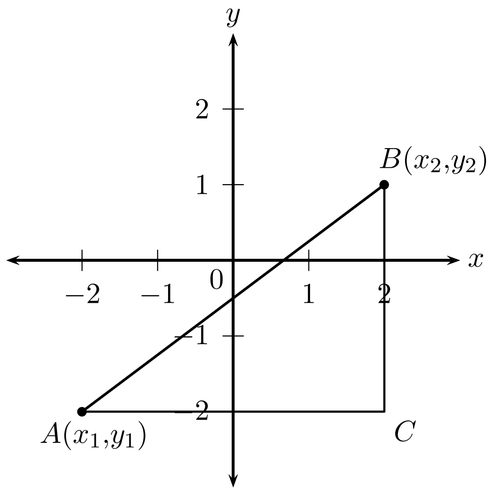
DEFINITION: Gradient

The gradient of a line is determined by the ratio of vertical change to horizontal change.

Gradient (m) describes the slope or steepness of the line joining two points. In the figure below, line OQ is the least steep and line OT is the steepest.



To derive the formula for gradient, we consider any right-angled triangle formed from A $(x_1; y_1)$ and B $(x_2; y_2)$ with hypotenuse AB, as shown in the diagram below. The gradient is determined by the ratio of the length of the vertical side of the triangle to the length of the horizontal side of the triangle. The length of the vertical side of the triangle is the difference in y-values of points A and B. The length of the horizontal side of the triangle is the difference in x-values of points A and B.



Therefore, gradient is determined using the following formula:

$$\text{Gradient } (m) = \frac{y_2 - y_1}{x_2 - x_1} \text{ or } \frac{y_1 - y_2}{x_1 - x_2}$$

Important: remember to be consistent: $m \neq \frac{y_1 - y_2}{x_2 - x_1}$

Video: VMbmr at www.everythingmaths.co.za

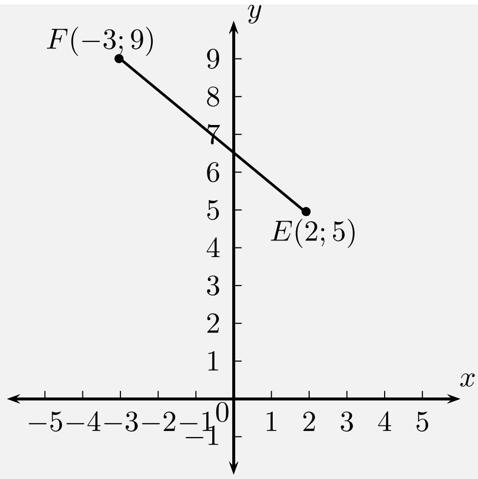
Example 3: *Gradient between two points*

QUESTION

Find the gradient of the line between points E$(2; 5)$ *and F*$(-3; 9)$.

SOLUTION

Step 1 : ***Draw a sketch***



Step 2 : ***Assign values to*** $(x_1; y_1)$ ***and*** $(x_2; y_2)$

Let the coordinates of E be $(x_1; y_1)$ and the coordinates of F be $(x_2; y_2)$.

$$x_1 = 2 \qquad y_1 = 5 \qquad x_2 = -3 \qquad y_2 = 9$$

Step 3 : ***Write down the formula for gradient***

$$m = \frac{y_2 - y_1}{x_2 - x_1}$$

Step 4 : ***Substitute known values***

$$\begin{aligned} m_{EF} &= \frac{9-5}{-3-2} \\ &= \frac{4}{-5} \end{aligned}$$

Step 5 : ***Write the final answer***

The gradient of $EF = -\frac{4}{5}$

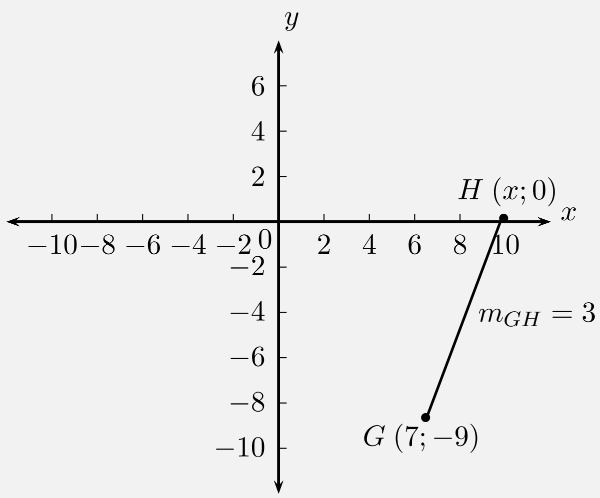
Example 4: *Gradient between two points*

QUESTION

Given $G(7;-9)$ and $H(x;0)$, with $m_{GH}=3$, Find x.

SOLUTION

Step 1 : ***Draw a sketch***



Step 2 : ***Assign values to** $(x_1;y_1)$ **and** $(x_2;y_2)$*

Let the coordinates of G be $(x_1;y_1)$ and the coordinates of H be $(x_2;y_2)$.

$$x_1=7 \qquad y_1=-9 \qquad x_2=x \qquad y_2=0$$

Step 3 : ***Write down the formula for gradient***

$$m=\frac{y_2-y_1}{x_2-x_1}$$

Step 4 : ***Substitute values and solve for** x*

$$\begin{aligned}
3 &= \frac{0-(-9)}{x-7} \\
3(x-7) &= 9 \\
x-7 &= \frac{9}{3} \\
x-7 &= 3 \\
x &= 3+7 \\
&= 10
\end{aligned}$$

Step 5 : ***Write the final answer***

The coordinates of H are $(10; 0)$.

Exercise 8 - 2

1. Find the gradient of AB if
   (a) $A(7; 10)$ and $B(-4; 1)$
   (b) $A(-5; -9)$ and $B(3; 2)$
   (c) $A(x-3; y)$ and $B(x; y+4)$
2. If the gradient of $CD = \frac{2}{3}$, find p given
   (a) $C(16; 2)$ and $D(8; p)$
   (b) $C(3; 2p)$ and $D(9; 14)$

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(1.) 00d0 (2.) 00d1

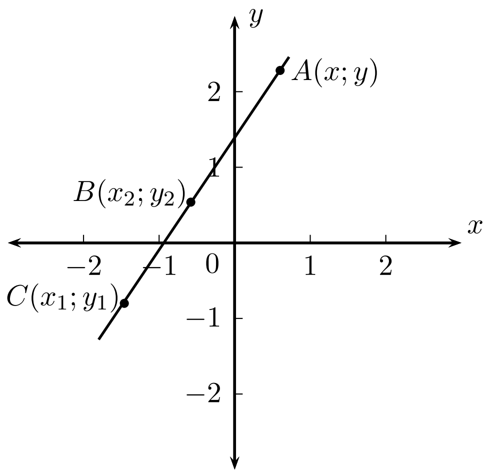
Straight lines

EMACY

DEFINITION: Straight line

A straight line is a set of points with a constant gradient between any two of the points.

Consider the diagram below with points $A(x;y)$, $B(x_2;y_2)$ and $C(x_1;y_1)$.



We have $m_{AB} = m_{BC} = m_{AC}$ and $m = \frac{y_2 - y_1}{x_2 - x_1} = \frac{y_1 - y_2}{x_1 - x_2}$

The general formula for finding the equation of a straight line is $\frac{y - y_1}{x - x_1} = \frac{y_2 - y_1}{x_2 - x_1}$ where $(x;y)$ is any point on the line.

This formula can also be written as $y - y_1 = m(x - x_1)$.

The standard form of the straight line equation is $y = mx + c$ where m is the gradient and c is the y-intercept.

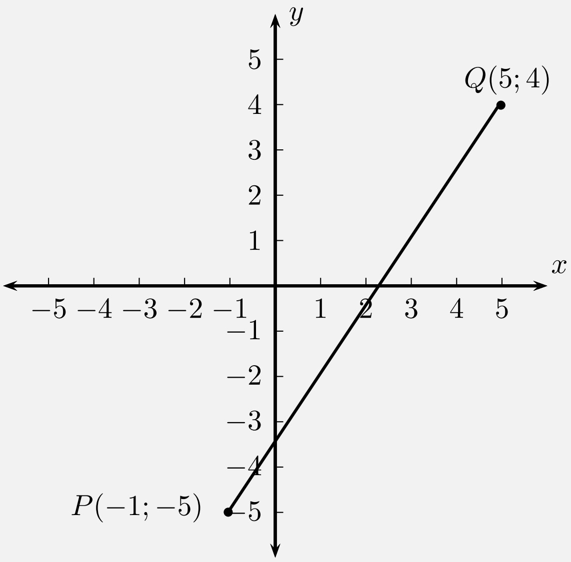
Example 5: *Finding the equation of a straight line*

QUESTION

Find the equation of the straight line through $P(-1;-5)$ and $Q(5;4)$.

SOLUTION

Step 1 : ***Draw a sketch***



Step 2 : ***Assign values***

Let the coordinates of P be $(x_1;y_1)$ and $Q(x_2;y_2)$

$$x_1 = -1 \qquad y_1 = -5 \qquad x_2 = 5 \qquad y_2 = 4$$

Step 3 : ***Write down the general formula of the line***

$$\frac{y - y_1}{x - x_1} = \frac{y_2 - y_1}{x_2 - x_1}$$

Step 4 : ***Substitute values and make y the subject of the equation***

$$\frac{y-(-5)}{x-(-1)} = \frac{4-(-5)}{5-(-1)}$$
$$\frac{y+5}{x+1} = \frac{3}{2}$$
$$2(y+5) = 3(x+1)$$
$$2y+10 = 3x+3$$
$$2y = 3x-7$$
$$y = \frac{3}{2}x-\frac{7}{2}$$

Step 5 : ***Write the final answer***

The equation of the straight line is $y = \frac{3}{2}x - \frac{7}{2}$.

Parallel and perpendicular lines *EMACZ*

Two lines that run parallel to each other are always the same distance apart and have equal gradients.

If two lines intersect perpendicularly, then the product of their gradients is equal to -1.

If line $WX \perp$ line YZ, then $m_{WX} \times m_{YZ} = -1$. Perpendicular lines have gradients that are the negative inverse of each other.

Video: VMbon at www.everythingmaths.co.za

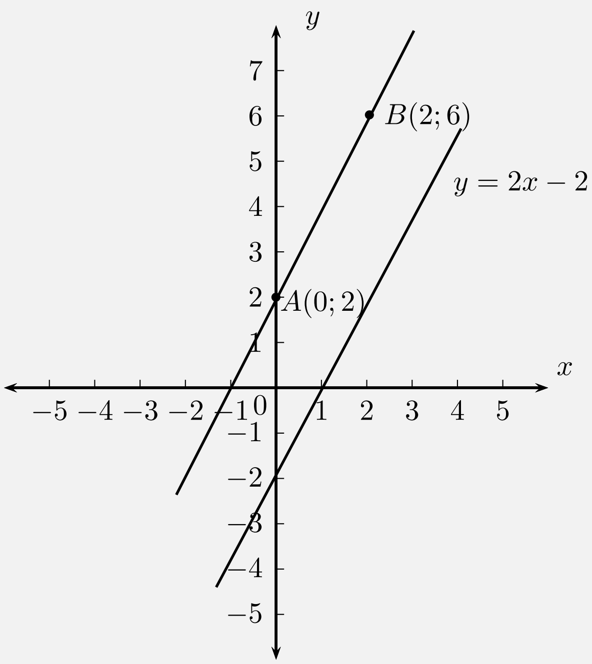
Example 6: *Parallel lines*

QUESTION

Prove that line AB with $A(0;2)$ and $B(2{,}6)$ is parallel to line $2x - y = 2$.

SOLUTION

Step 1 : ***Draw a sketch***



(Be careful - some lines may look parallel but are not!)

Step 2 : ***Write down the formula for gradient***

$$m = \frac{y_2 - y_1}{x_2 - x_1}$$

Step 3 : ***Substitute values to find the gradient for line*** AB

$$\begin{aligned} m_{AB} &= \frac{6-2}{2-0} \\ &= \frac{4}{2} \\ &= 2 \end{aligned}$$

Step 4 : ***Check that the equation of CD is in the standard form $y = mx + c$***

$$\begin{aligned} 2x - y &= 2 \\ y &= 2x - 2 \\ \therefore m_{CD} &= 2 \end{aligned}$$

Step 5 : ***Write the final answer***

$$m_{AB} = m_{CD}$$

therefore line AB is parallel to cd, $y = 2x - 2$.

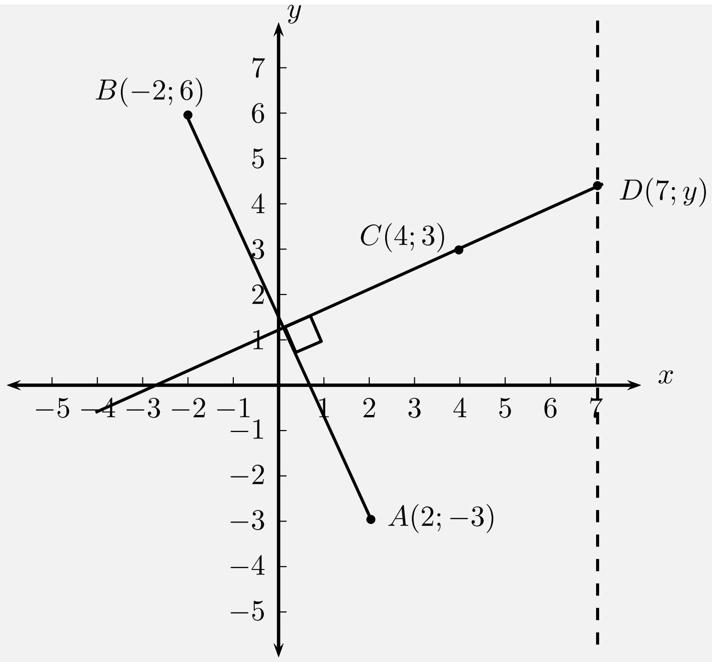
Example 7: *Perpendicular lines*

QUESTION

Line AB is perpendicular to line CD. Find y given $A(2;-3)$, $B(-2;6)$, $C(4;3)$ and $D(7;y)$.

SOLUTION

Step 1 : ***Draw a sketch***



Step 2 : ***Write down the relationship between the gradients of the perpendicular lines*** $AB \perp CD$

$$m_{AB} \times m_{CD} = -1$$

$$\frac{y_B - y_A}{x_B - x_A} \times \frac{y_D - y_C}{x_D - x_C} = -1$$

Step 3 : ***Substitute values and solve for*** y

$$\begin{aligned}
\frac{6-(-3)}{-2-2} \times \frac{y-3}{7-4} &= -1 \\
\frac{9}{-4} \times \frac{y-3}{3} &= -1 \\
\frac{y-3}{3} &= -1 \times \frac{-4}{9} \\
\frac{y-3}{3} &= \frac{4}{9} \\
y-3 &= \frac{4}{9} \times 3 \\
y-3 &= \frac{4}{3}
\end{aligned}$$

$$\begin{aligned} y &= \frac{4}{3} + 3 \\ &= \frac{4+9}{3} \\ &= \frac{13}{3} \\ &= 4\frac{1}{3} \end{aligned}$$

Step 4 : ***Write the final answer***

Therefore the coordinates of D are $(7; 4\frac{1}{3})$.

Horizontal and vertical lines — *EMADA*

A line that runs parallel to the x-axis is called a horizontal line and has a gradient of zero. This is because there is no vertical change:

$$m = \frac{\text{change in } y}{\text{change in } x} = \frac{0}{\text{change in } x} = 0$$

A line that runs parallel to the y-axis is called a vertical line and its gradient is undefined. This is because there is no horizontal change:

$$m = \frac{\text{change in } y}{\text{change in } x} = \frac{\text{change in } y}{0} = \text{ undefined}$$

Points on a line — *EMADB*

A straight line is a set of points with a constant gradient between any of the two points. There are two methods to prove that points lie on the same line; the gradient method and a longer method using the distance formula.

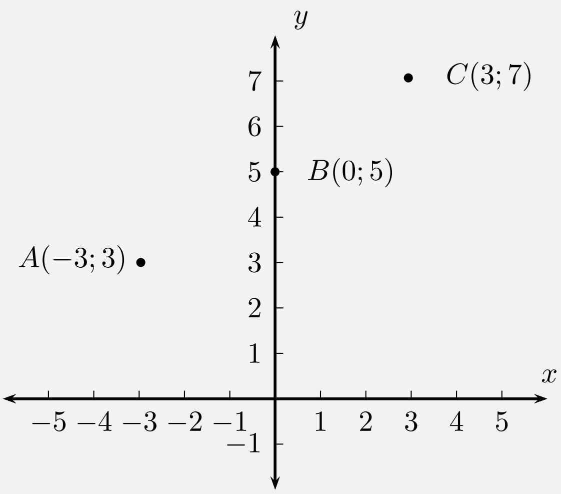
Example 8: *Points on a line*

QUESTION

Prove that $A(-3;3)$, $B(0;5)$ and $C(3;7)$ are on a straight line.

SOLUTION

Step 1 : ***Draw a sketch***



Step 2 : ***Calculate two gradients between any of the three points***

$$m = \frac{y_2 - y_1}{x_2 - x_1}$$

$$m_{AB} = \frac{5-3}{0-(-3)} = \frac{2}{3}$$

and

$$m_{BC} = \frac{7-5}{3-0} = \frac{2}{3}$$

OR

$$m_{AC} = \frac{3-7}{3-3} = \frac{-4}{-6} = \frac{2}{3}$$

and

$$m_{BC} = \frac{7-5}{3-0} = \frac{2}{3}$$

Step 3 : ***Explain your answer***

$$m_{AB} = m_{BC} = m_{AC}$$

Therefore the points A, B and C are on a straight line.

To prove that three points are on a straight line using the distance formula, we must calculate the distances between each pair of points and then prove that the sum of the two smaller distances equals the longest distance.

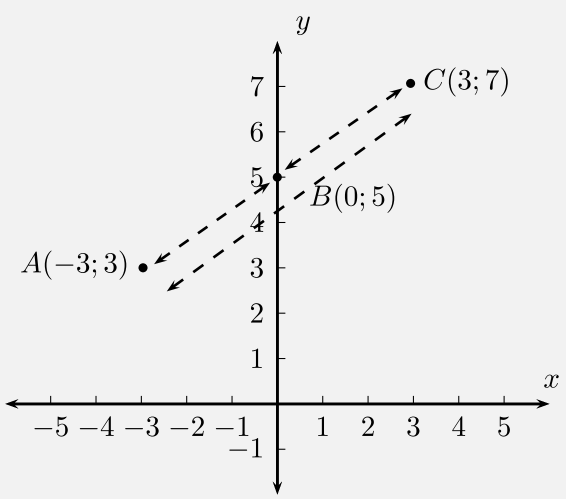
Example 9: *Points on a straight line*

QUESTION

Prove that $A(-3;3)$, $B(0;5)$ and $C(3;7)$ are on a straight line.

SOLUTION

Step 1 : ***Draw a sketch***



Step 2 : ***Calculate the three distances AB, BC and AC***

$$d_{AB} = \sqrt{(-3-0)^2 + (3-5)^2} = \sqrt{(-3)^2 + (-2)^2} = \sqrt{9+4} = \sqrt{13}$$
$$d_{BC} = \sqrt{(0-3)^2 + (5-7)^2} = \sqrt{(-3)^2 + (-2)^2} = \sqrt{9+4} = \sqrt{13}$$
$$d_{AC} = \sqrt{((-3)-3)^2 + (3-7)^2} = \sqrt{(-6)^2 + (-4)^2} = \sqrt{36+16} = \sqrt{52}$$

Step 3 : ***Find the sum of the two shorter distances***

$$d_{AB} + d_{BC} = \sqrt{13} + \sqrt{13} = 2\sqrt{13} = \sqrt{4 \times 13} = \sqrt{52}$$

Step 4 : ***Explain your answer***

$$d_{AB} + d_{BC} = d_{AC}$$

therefore points A, B and C lie on the same straight line.

Exercise 8 - 3

1. Determine whether AB and CD are parallel, perpendicular or neither if:
   (a) $A(3;-4)$, $B(5;2)$, $C(-1;-1)$, $D(7;23)$
   (b) $A(3;-4)$, $B(5;2)$, $C(-1;-1)$, $D(0;-4)$
   (c) $A(3;-4)$, $B(5;2)$, $C(-1;-1)$, $D(1;2)$
2. Determine whether the following points lie on the same straight line:
   (a) $E(0;3)$, $F(-2;5)$, $G(2;1)$
   (b) $H(-3;-5)$, $I(-0;0)$, $J(6;10)$
   (c) $K(-6;2)$, $L(-3;1)$, $M(1;-1)$
3. Points $P(-6;2)$, $Q(2;-2)$ and $R(-3;r)$ lie on a straight line. Find the value of r.
4. Line PQ with $P(-1;-7)$ and $Q(q;0)$ has a gradient of 1. Find q.

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(1.) 00d2 (2.) 00d3 (3.) 00d4 (4.) 00d5

8.4 Mid-point of a line

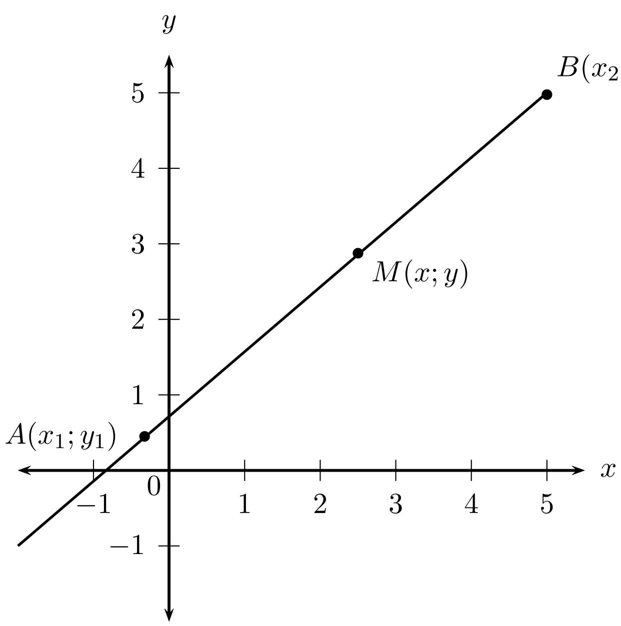
EMADC

Investigation:	*Finding the mid-point of a line*

On graph paper, accurately plot the points $P(2; 1)$ and $Q(-2; 2)$ and draw the line PQ.

- Fold the piece of paper so that point P is exactly on top of point Q.
- Where the folded line intersects with line PQ, label point S.
- Count the blocks and find the exact position of S.
- Write down the coordinates of S.

To calculate the coordinates of the mid-point $M(x; y)$ of any line between two points $A(x_1; y_1)$ and $B(x_2; y_2)$ we use the following formulae:



$$
\begin{aligned}
x &= \frac{x_1 + x_2}{2} \\
y &= \frac{y_1 + y_2}{2}
\end{aligned}
$$

From this we obtain the mid-point of a line:

$$\text{Midpoint } M(x;y) = \left(\frac{x_1 + x_2}{2}; \frac{y_1 + y_2}{2}\right)$$

Video: VMbpr at www.everythingmaths.co.za

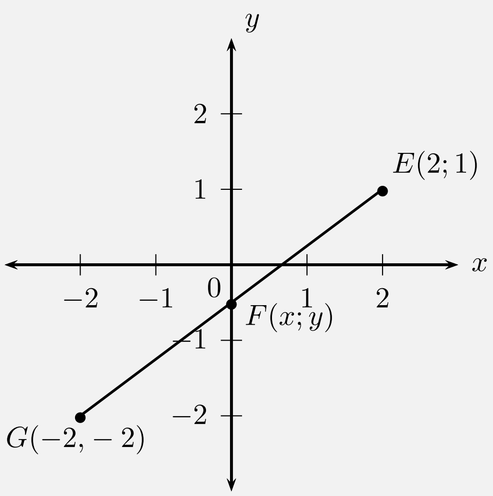
Example 10: *Calculating the mid-point*

QUESTION

Calculate the coordinates of the mid-point $F(x;y)$ of the line segment between point $E(2;1)$ and point $G(-2;-2)$.

SOLUTION

Step 1 : ***Draw a sketch***



Step 2 : ***Assign values to*** $(x_1; y_1)$ ***and*** $(x_2; y_2)$

$$x_1 = -2 \qquad y_1 = -2 \qquad x_1 = 2 \qquad y_2 = 1$$

Step 3 : ***Write down the mid-point formula***

$$F(x;y) = (\frac{x_1 + x_2}{2}; \frac{y_1 + y_2}{2})$$

Step 4 : ***Substitute values into the mid-point formula***

$$\begin{aligned} x &= \frac{x_1 + x_2}{2} \\ &= \frac{-2+2}{2} \\ &= 0 \\ y &= \frac{y_1 + y_2}{2} \\ &= \frac{-2+1}{2} \\ &= -\frac{1}{2} \end{aligned}$$

Step 5 : ***Write the answer***

The mid-point is at $F(0; -\frac{1}{2})$

Step 6 : ***Confirm answer using distance formula***

Using the distance formula, we can confirm that the distances from the mid-point to each end point are equal:

$$\begin{aligned} PS &= \sqrt{(x_1 - x_2)^2 + (y_1 - y_2)^2} \\ &= \sqrt{(0-2)^2 + (-0{,}5-1)^2} \\ &= \sqrt{(-2)^2 + (-1{,}5)^2} \\ &= \sqrt{4 + 2{,}25} \\ &= \sqrt{6{,}25} \end{aligned}$$

and

$$\begin{aligned} QS &= \sqrt{(x_1 - x_2)^2 + (y_1 - y_2)^2} \\ &= \sqrt{(0 - (-2))^2 + (-0,5 - (-2))^2} \\ &= \sqrt{(0 + 2)^2 + (-0,5 + 2)^2} \\ &= \sqrt{(2)^2 + (-1,5)^2} \\ &= \sqrt{4 + 2,25} \\ &= \sqrt{6,25} \end{aligned}$$

As expected, $PS = QS$, therefore F is the mid-point.

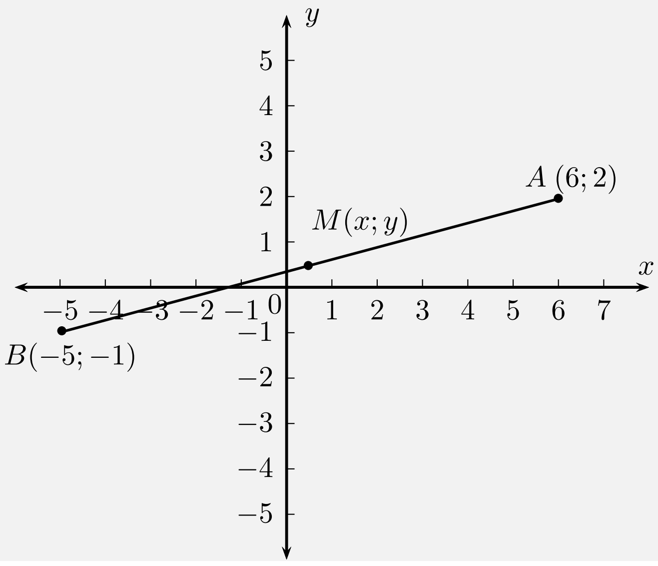
Example 11: *Calculating the mid-point*

QUESTION

Find the mid-point of line AB, given $A(6;2)$ and $B(-5;-1)$.

SOLUTION

Step 1 : **Draw a sketch**



From the sketch, we can estimate that M will lie in quadrant I, with positive x- and y-coordinates.

Step 2 : ***Assign values to*** $(x_1; y_1)$ ***and*** $(x_2; y_2)$

Let the mid-point be $M(x; y)$

$$x_1 = 6 \qquad y_1 = 2 \qquad x_2 = -5 \qquad y_2 = -1$$

Step 3 : ***Write down the mid-point formula***

$$M(x; y) = \left(\frac{x_1 + x_2}{2}; \frac{y_1 + y_2}{2}\right)$$

Step 4 : ***Substitute values and simplify***

$$M(x; y) = \left(\frac{6 - 5}{2}; \frac{2 - 1}{2}\right) = \left(\frac{1}{2}; \frac{1}{2}\right)$$

Step 5 : ***Write the final answer***

$M(\frac{1}{2}; \frac{1}{2})$ is the mid-point of line AB.

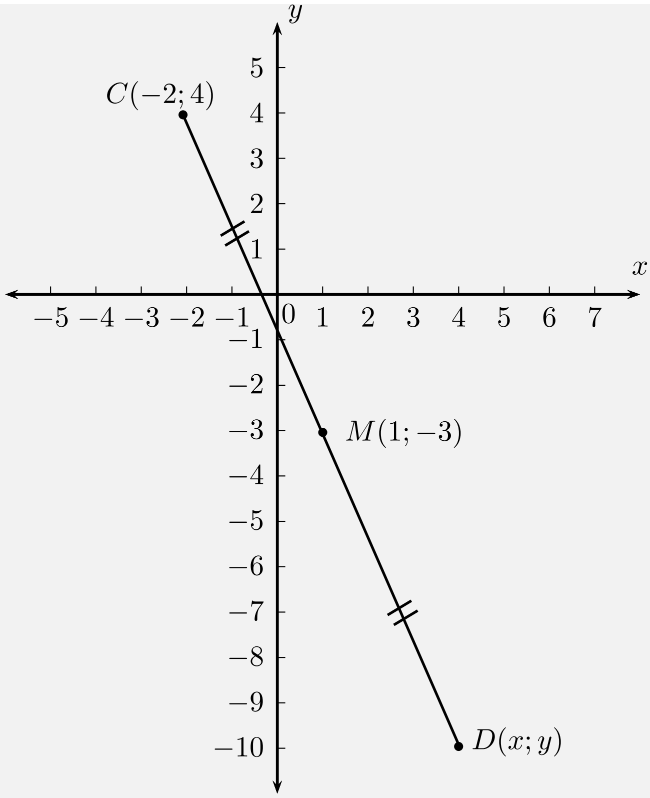
Example 12: *Using the mid-point formula*

QUESTION

The line joining $C(-2; 4)$ *and* $D(x; y)$ *has the mid-point* $M(1; -3)$. *Find point* D.

SOLUTION

Step 1 : ***Draw a sketch***



From the sketch, we can estimate that D will lie in Quadrant IV, with a positive x- and negative y-coordinate.

Step 2 : ***Assign values to*** $(x_1; y_1)$ ***and*** $(x_2; y_2)$

Let the coordinates of C be $(x_1; y_1)$ and the coordinates of D be $(x_2; y_2)$.

$$x_1 = -2 \qquad y_1 = 4 \qquad x_2 = x \qquad y_2 = y$$

Step 3 : ***Write down the mid-point formula***

$$M(x; y) = \left(\frac{x_1 + x_2}{2}; \frac{y_1 + y_2}{2}\right)$$

Step 4 : ***Substitute values and solve for*** x_2 ***and*** y_2

$$\begin{aligned} 1 &= \frac{-2 + x_2}{2} & -3 &= \frac{4 + y_2}{2} \\ 1 \times 2 &= -2 + x_2 & -3 \times 2 &= 4 + y_2 \\ 2 &= -2 + x_2 & -6 &= 4 + y_2 \\ x_2 &= 2 + 2 & y_2 &= -6 - 4 \end{aligned}$$

$$x_2 = 4 \qquad y_2 = -10$$

Step 5 : ***Write the final answer***

The coordinates of point D are $(4; -10)$.

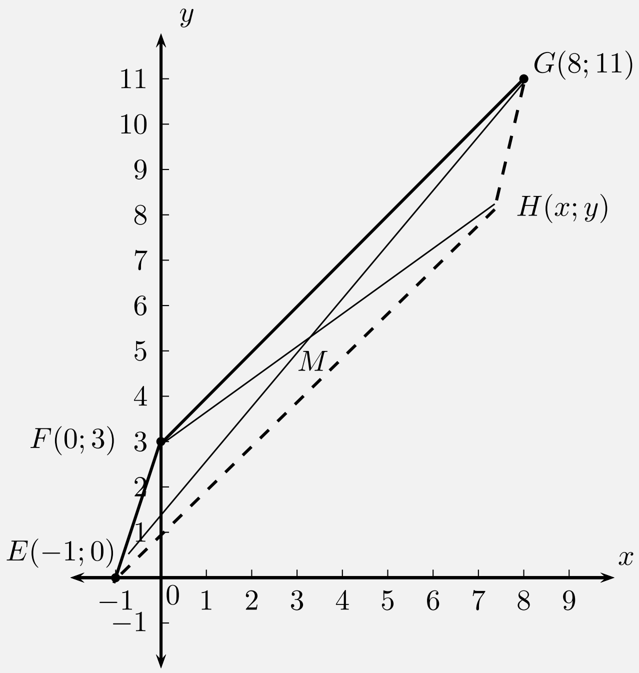
Example 13: *Using the mid-point formula*

QUESTION

Points $E(-1; 0)$, $F(0; 3)$, $G(8; 11)$ and $H(x; y)$ are points on the Cartesian plane. Find $H(x; y)$ if $EFGH$ is a parallelogram.

SOLUTION

Step 1 : ***Draw a sketch***



Method: the diagonals of a parallelogram bisect each other, therefore the mid-point of EG will be the same as the mid-point of FH. We must first find the mid-point of EG. We can then use it to determine the coordinates of point H.

Step 2 : ***Assign values to*** $(x_1; y_1)$ ***and*** $(x_2; y_2)$

Let the mid-point of EG be $M(x; y)$

$$x_1 = -1 \qquad y_1 = 0 \qquad x_2 = 8 \qquad y_2 = 11$$

Step 3 : ***Write down the mid-point formula***

$$M(x; y) = \left(\frac{x_1 + x_2}{2}; \frac{y_1 + y_2}{2}\right)$$

Step 4 : ***Substitute values calculate the coordinates of*** M

$$M(x; y) = \left(\frac{-1 + 8}{2}; \frac{0 + 11}{2}\right) = \left(\frac{7}{2}; \frac{11}{2}\right)$$

Step 5 : ***Use the coordinates of*** M ***to determine*** H

M is also the mid-point of FH so we use $M(\frac{7}{2}; \frac{11}{2})$ and $F(0; 3)$ to solve for $H(x; y)$

Step 6 : ***Substitute values and solve for*** x ***and*** y

$$M\left(\frac{7}{2}; \frac{11}{2}\right) = \left(\frac{x_1 + x_2}{2}; \frac{y_1 + y_2}{2}\right)$$

$$\begin{aligned} \frac{7}{2} &= \frac{0 + x}{2} & \qquad \frac{11}{2} &= \frac{3 + y}{2} \\ 7 &= x + 0 & 11 &= 3 + y \\ x &= 7 & y &= 8 \end{aligned}$$

Step 7 : ***Write the final answer***

The coordinates of H are $(7; 8)$.

Exercise 8 - 4

1. Find the mid-points of the following lines:

(a) $A(2; 5)$, $B(-4; 7)$

(b) $C(5; 9)$, $D(23; 55)$

(c) $E(x + 2; y - 1)$, $F(x - 5; y - 4)$

2. The mid-point M of PQ is $(3; 9)$. Find P if Q is $(-2; 5)$.

3. $PQRS$ is a parallelogram with the points $P(5;3)$, $Q(2;1)$ and $R(7;-3)$. Find point S.

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(1.) 00d6 (2.) 00d7 (3.) 00d8

Chapter 8 | Summary

Summary presentation: at www.everythingmaths.co.za

- A point is an ordered pair of numbers written as $(x;y)$.
- Distance is a measure of the length between two points.
- The formula for finding the distance between any two points is:

$$d = \sqrt{(x_1 - x_2)^2 + (y_1 - y_2)^2}$$

- The gradient between two points is determined by the ratio of vertical change to horizontal change.
- The formula for finding the gradient of a line is:

$$m = \frac{y_2 - y_1}{x_2 - x_1}$$

- A straight line is a set of points with a constant gradient between any two of the points.
- The standard form of the straight line equation is $y = mx + c$.
- The equation of a straight line can also be written as

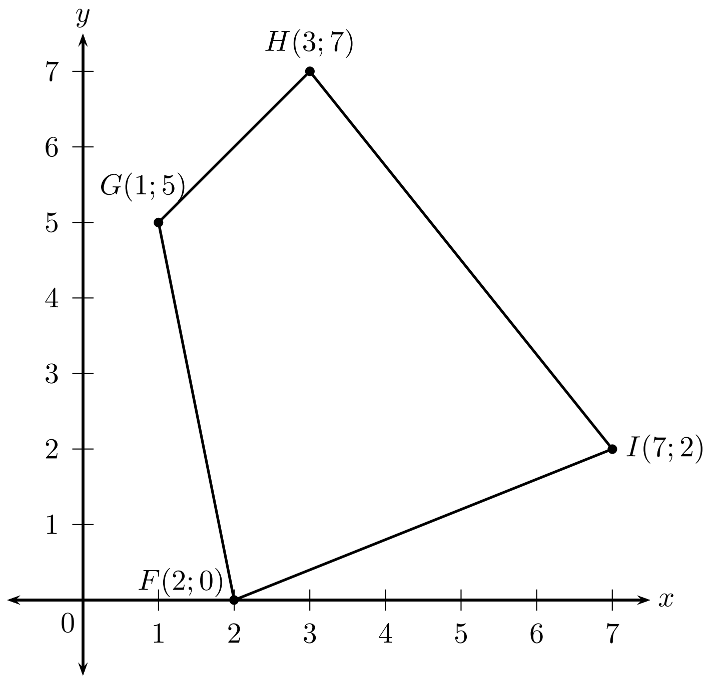
$$\frac{y - y_1}{x - x_1} = \frac{y_2 - y_1}{x_2 - x_1}$$

- If two lines are parallel, their gradients are equal.
- If two lines are perpendicular, the product of their gradients is equal to -1.
- For horizontal lines the gradient is equal to 0.
- For vertical lines the gradient is undefined.
- The formula for finding the mid-point between two points is:

$$M(x;y) = \left(\frac{x_1 + x_2}{2}; \frac{y_1 + y_2}{2}\right)$$

Chapter 8 End of Chapter Exercises

1. Represent the following figures in the Cartesian plane:
   (a) Triangle DEF with $D(1;2)$, $E(3;2)$ and $F(2;4)$
   (b) Quadrilateral $GHIJ$ with $G(2;-1)$, $H(0;2)$, $I(-2;-2)$ and $J(1;-3)$
   (c) Quadrilateral $MNOP$ with $M(1;1)$, $N(-1;3)$, $O(-2;3)$ and $P(-4;1)$
   (d) Quadrilateral $WXYZ$ with $W(1;-2)$, $X(-1;-3)$, $Y(2;-4)$ and $Z(3;-2)$
2. In the diagram below, the vertices of the quadrilateral are $F(2;0)$, $G(1;5)$, $H(3;7)$ and $I(7;2)$.



   (a) Calculate the lengths of the sides of $FGHI$.
   (b) Are the opposite sides of $FGHI$ parallel?
   (c) Do the diagonals of $FGHI$ bisect each other?
   (d) Can you state what type of quadrilateral $FGHI$ is? Give reasons for your answer.
3. Consider a quadrilateral $ABCD$ with vertices $A(3;2)$, $B(4;5)$, $C(1;7)$ and $D(1;3)$.
   (a) Draw the quadrilateral.
   (b) Find the lengths of the sides of the quadrilateral.
4. $ABCD$ is a quadrilateral with vertices $A(0;3)$, $B(4;3)$, $C(5;-1)$ and $D(-1;-1)$.

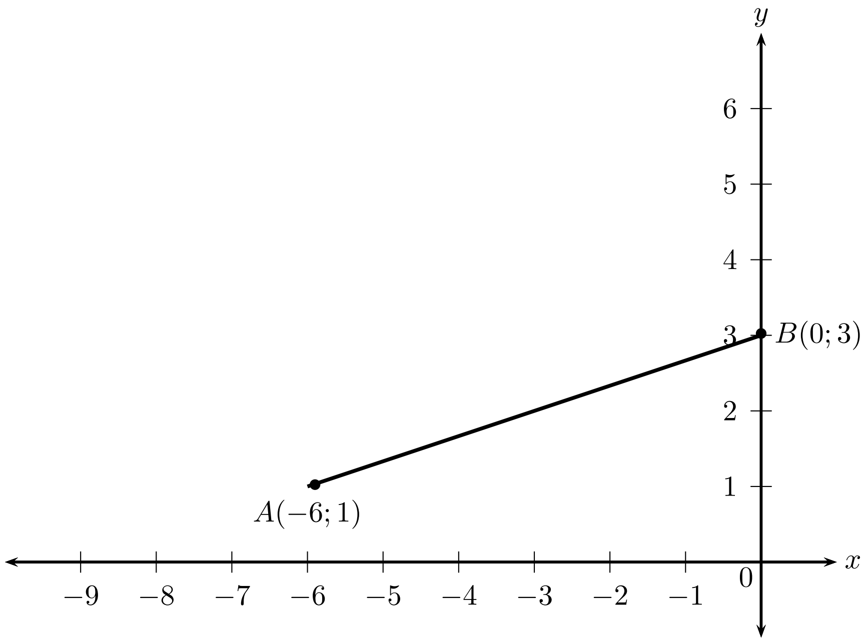
(a) Show by calculation that:
   i. $AD = BC$
   ii. $AB \parallel DC$

(b) What type of quadrilateral is $ABCD$?

(c) Show that the diagonals AC and BD do not bisect each other.

5. P, Q, R and S are the points $(-2;0)$, $(2;3)$, $(5;3)$, $(-3;-3)$ respectively.
   (a) Show that:
      i. $SR = 2PQ$
      ii. $SR \parallel PQ$
   (b) Calculate:
      i. PS
      ii. QR
   (c) What kind of quadrilateral is $PQRS$? Give reasons for your answer.
6. $EFGH$ is a parallelogram with vertices $E(-1;2)$, $F(-2;-1)$ and $G(2;0)$. Find the coordinates of H by using the fact that the diagonals of a parallelogram bisect each other.
7. $PQRS$ is a quadrilateral with points $P(0;-3)$, $Q(-2;5)$, $R(3;2)$ and $S(3;-2)$ in the Cartesian plane.
   (a) Find the length of QR.
   (b) Find the gradient of PS.
   (c) Find the mid-point of PR.
   (d) Is $PQRS$ a parallelogram? Give reasons for your answer.
8. $A(-2;3)$ and $B(2;6)$ are points in the Cartesian plane. $C(a;b)$ is the mid-point of AB. Find the values of a and b.
9. Consider triangle ABC with vertices $A(1;3)$, $B(4;1)$ and $C(6;4)$.
   (a) Sketch triangle ABC on the Cartesian plane.
   (b) Show that ABC is an isosceles triangle.
   (c) Determine the coordinates of M, the mid-point of AC.
   (d) Determine the gradient of AB.
   (e) Show that $D(7;-1)$ lies on the line that goes through A and B.

10. In the diagram, A is the point $(-6;1)$ and B is the point $(0;3)$



(a) Find the equation of line AB.

(b) Calculate the length of AB.

11. $\triangle PQR$ has vertices $P(1;8)$, $Q(8;7)$ and $R(7;0)$. Show through calculation that $\triangle PQR$ is a right angled isosceles triangle.

12. $\triangle ABC$ has vertices $A(-3;4)$, $B(3;-2)$ and $R(-5;-2)$. M is the midpoint of AC and N the midpoint of BC. Use $\triangle ABC$ to prove the midpoint theorem using analytical geometrical methods.

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(1.) 00d9 (2.) 00da (3.) 00db (4.) 00dc (5.) 00dd

(6.) 00de (7.) 00df (8.) 00dg (9.) 00dh (10.) 00di

(11.) 022r (12.) 022s

Statistics 9

When running an experiment or conducting a survey we can potentially end up with many hundreds, thousands or even millions of values in the resulting data set. Too many data can be overwhelming and we need to reduce them or represent them in a way that is easier to understand and communicate.

Statistics is about summarising data. The methods of statistics allow us to represent the essential information in a data set while disregarding the unimportant information. We have to be careful to make sure that we do not accidentally throw away some of the important aspects of a data set.

By applying statistics properly we can highlight the important aspects of data and make the data easier to interpret. By applying statistics poorly or dishonestly we can also hide important information and let people draw the wrong conclusions.

In this chapter we will look at a few numerical and graphical ways in which data sets can be represented, to make them easier to interpret.

Video: VMbqd at www.everythingmaths.co.za

9.1 Collecting data

EMADD

> *DEFINITION: Data*
>
> Data refer to the pieces of information that have been observed and recorded, from an experiment or a survey.

We distinguish between two main types of data: quantitative and qualitative. Note that the word *data* is the plural of the word *datum*, and therefore one should say, "the data are" and not "the data is".

DEFINITION: Quantitative data

Quantitative data are data that can be written as numbers.

Quantitative data can be discrete or continuous. Discrete quantitative data can be represented by integers and usually occur when we count things, for example, the number of learners in a class, the number of molecules in a chemical solution, or the number of SMS messages sent in one day.

Continuous quantitative data can be represented by real numbers, for example, the height or mass of a person, the distance travelled by a car, or the duration of a phone call.

DEFINITION: Qualitative data

Qualitative data are data that cannot be written as numbers.

Two common types of qualitative data are categorical and anecdotal data. Categorical data can come from one of a limited number of possibilities, for example, your favourite cool drink, the colour of your cell phone, or the language that you learned to speak at home.

Anecdotal data take the form of an interview or a story, for example, when you ask someone what their personal experience was when using a product, or what they think of someone else's behaviour.

Categorical qualitative data are sometimes turned into quantitative data by counting the number of times that each category appears. For example, in a class with 30 learners, we ask everyone what the colours of their cell phones are and get the following responses:

black	black	black	white	purple	red	red	black	black	black
white	white	black	black	black	black	purple	black	black	white
purple	black	red	red	white	black	orange	orange	black	white

This is a categorical qualitative data set since each of the responses comes from one of a small number of possible colours. We can represent exactly the same data in a different way, by counting how many times each colour appears.

Colour	Count
black	15
white	6
red	4
purple	3
orange	2

This is a discrete quantitative data set since each count is an integer.

Example 1: *Qualitative and quantitative data*

QUESTION

Thembisile is interested in becoming an airtime reseller to his classmates. He would like to know how much business he can expect from them. He asked each of his 20 classmates how many SMS messages they sent during the previous day. The results were:

20	3	0	14	30	9	11	13	13	15
9	13	16	12	13	7	17	14	9	13

Is this data set qualitative or quantitative? Explain your answer.

SOLUTION

The number of SMS messages is a count represented by an integer, which means that it is quantitative and discrete.

Example 2: *Qualitative and quantitative data*

QUESTION

Thembisile would like to know who the most popular cellular provider is among learners in his school. This time Thembisile randomly selects 20 *learners from the entire school and asks them which cellular provider they currently use. The results were:*

Cell C	*Vodacom*	*Vodacom*	*MTN*	*Vodacom*
MTN	*MTN*	*Virgin Mobile*	*Cell C*	*8-ta*
Vodacom	*MTN*	*Vodacom*	*Vodacom*	*MTN*
Vodacom	*Vodacom*	*Vodacom*	*Virgin Mobile*	*MTN*

Is this data set qualitative or quantitative? Explain your answer.

SOLUTION

Since each response is not a number, but one of a small number of possibilities, these are categorical qualitative data.

9.2 Measures of central tendency EMADE

Mean EMADF

DEFINITION: Mean

The mean is the sum of a set of values, divided by the number of values in the set. The notation for the mean of a set of values is a horizontal bar over the variable used to represent the set. The formula for the mean of a data set $\{x_1, x_2, \ldots, x_n\}$ is

$$\begin{aligned}\overline{x} &= \frac{1}{n}\sum_{i=1}^{n} x_i \\ &= \frac{x_1 + x_2 + \cdots + x_n}{n}\end{aligned}$$

The mean is sometimes also called the average or the arithmetic mean.

Video: VMbqd at www.everythingmaths.co.za

Example 3: *Calculating the mean*

QUESTION

What is the mean of the data set $\{10;\ 20;\ 30;\ 40;\ 50\}$*?*

SOLUTION

Step 1 : ***Calculate the sum of the data***

$$10 + 20 + 30 + 40 + 50 = 150$$

Step 2 : ***Divide by the number of values in the data set to get the mean***

Since there are 5 values in the data set, the mean is

$$\text{Mean} = \frac{150}{5} = 30$$

Median

EMADG

DEFINITION: Median

The median of a data set is the value in the central position, when the data set has been arranged from the lowest to the highest value.

Note that exactly half of the values from the data set are less than the median and the other half are greater than the median.

To calculate the median of a quantitative data set, first sort the data from the smallest to the largest value and then find the value in the middle. If there are an odd number of data, the median will be equal to one of the values in the data set. If there are an even number of data, the median will lie halfway between two values in the data set.

Example 4: *Median for an odd number of values*

QUESTION

What is the median of $\{10;\ 14;\ 86;\ 2;\ 68;\ 99;\ 1\}$*?*

SOLUTION

Step 1 : ***Sort the values***

The values in the data set, arranged from the smallest to the largest, are

$$1;\ 2;\ 10;\ 14;\ 68;\ 86;\ 99$$

Step 2 : ***Find the number in the middle***

There are 7 values in the data set. Since there are an odd number of values, the median will be equal to the value in the middle, namely, in the 4^{th} position. Therefore the median of the data set is 14.

Example 5: *Median for an even number of values*

QUESTION

What is the median of $\{11;\ 10;\ 14;\ 86;\ 2;\ 68;\ 99;\ 1\}$*?*

SOLUTION

Step 1 : ***Sort the values***

The values in the data set, arranged from the smallest to the largest, are

$$1;\ 2;\ 10;\ 11;\ 14;\ 68;\ 86;\ 99$$

Step 2 : ***Find the number in the middle***

There are 8 values in the data set. Since there are an even number of values, the median will be halfway between the two values in the middle, namely, between the 4^{th} and 5^{th} positions. The value in the 4^{th} position is 11 and the value in the 5^{th} position is 14. The median lies halfway between these two values and is therefore

$$\text{Median} = \frac{11+14}{2} = 12{,}5$$

Mode

EMADH

DEFINITION: Mode

The mode of a data set is the value that occurs most often in the set. The mode can also be described as the most frequent or most common value in the data set.

To calculate the mode, we simply count the number of times that each value appears in the data set and then find the value that appears most often.

A data set can have more than one mode if there is more than one value with the highest count. For example, both 2 and 3 are modes in the data set $\{1;\ 2;\ 2;\ 3;\ 3\}$. If all points in a data set occur with equal frequency, it is equally accurate to describe the data set as having many modes or no mode.

Example 6: *Finding the mode*

QUESTION

Find the mode of the data set $\{2;\ 2;\ 3;\ 4;\ 4;\ 4;\ 6;\ 6;\ 7;\ 8;\ 8;\ 10;\ 10\}$.

SOLUTION

Step 1 : ***Count the number of times that each value appears in the data set***

Value	Count
2	2
3	1
4	3
6	2
7	1
8	2
10	2

Step 2 : ***Find the value that appears most often***

From the table above we can see that 4 is the only value that appears 3 times, and all the other values appear less often. Therefore the mode of the data set is 4.

One problem with using the mode as a measure of central tendency is that we can usually not compute the mode of a continuous data set. Since continuous values can lie anywhere on the real line, any particular value will almost never repeat. This means that the frequency of each value in the data set will be 1 and that there will be no mode. We will look at one way of addressing this problem in the section on grouping data.

Example 7: *Comparison of measures of central tendency*

QUESTION

There are regulations in South Africa related to bread production to protect consumers. By law, if a loaf of bread is not labelled, it must weigh 800 g, with the leeway of 5 percent under or 10 percent over. Vishnu is interested in how a well-known, national retailer measures up to this standard. He visited his local branch of the supplier and recorded the masses of 10 different loaves of bread for one week. The results, in grams, are given below:

Monday	***Tuesday***	***Wednesday***	***Thursday***	***Friday***	***Saturday***	***Sunday***
802,4	787,8	815,7	807,4	801,5	786,6	799,0
796,8	798,9	809,7	798,7	818,3	789,1	806,0
802,5	793,6	785,4	809,3	787,7	801,5	799,4
819,6	812,6	809,1	791,1	805,3	817,8	801,0
801,2	795,9	795,2	820,4	806,6	819,5	796,7
789,0	796,3	787,9	799,8	789,5	802,1	802,2
789,0	797,7	776,7	790,7	803,2	801,2	807,3
808,8	780,4	812,6	801,8	784,7	792,2	809,8
802,4	790,8	792,4	789,2	815,6	799,4	791,2
796,2	817,6	799,1	826,0	807,9	806,7	780,2

1. *Is this data set qualitative or quantitative? Explain your answer.*
2. *Determine the mean, median and mode of the mass of a loaf of bread for each day of the week. Give your answer correct to 1 decimal place.*
3. *Based on the data, do you think that this supplier is providing bread within the South African regulations?*

SOLUTION

Step 1 : ***Qualitative or quantitative?***

Since each mass can be represented by a number, the data set is quantitative. Furthermore, since a mass can be any real number, the data are continuous.

Step 2 : ***Calculate the mean***

In each column (for each day of the week), we add up the measurements and divide by the number of measurements, 10. For Monday, the sum of the measured values is 8007,9 and so the mean for Monday is

$$\frac{8007{,}9}{10} = 800{,}8 \text{ g}$$

In the same way, we can compute the mean for each day of the week. See the table below for the results.

Step 3 : ***Calculate the median***

In each column we sort the numbers from lowest to highest and find the value in the middle. Since there are an even number of measurements (10), the median is halfway between the two numbers in the middle. For Monday, the sorted list of numbers is

$$789{,}0;\ 789{,}0;\ 796{,}2;\ 796{,}7;\ 801{,}2;$$
$$802{,}3;\ 802{,}3;\ 802{,}5;\ 808{,}7;\ 819{,}6$$

The two numbers in the middle are $801{,}2$ and $802{,}3$ and so the median is

$$\frac{801{,}2 + 802{,}3}{2} = 801{,}8 \text{ g}$$

In the same way, we can compute the median for each day of the week:

Day	**Mean**	**Median**
Monday	$800{,}8$ g	$801{,}8$ g
Tuesday	$797{,}2$ g	$796{,}1$ g
Wednesday	$798{,}4$ g	$797{,}2$ g
Thursday	$803{,}4$ g	$800{,}8$ g
Friday	$802{,}0$ g	$804{,}3$ g
Saturday	$801{,}6$ g	$801{,}4$ g
Sunday	$799{,}3$ g	$800{,}2$ g

From the above calculations we can see that the means and medians are close to one another, but not quite equal. In the next worked example we will see that the mean and median are not always close to each other.

Step 4 : ***Determine the mode***

Since the data are continuous we cannot compute the mode. In the next section we will see how we can group data in order to make it possible to compute an approximation for the mode.

Step 5 : ***Conclusion: Is the supplier reliable?***

From the question, the requirements are that the mass of a loaf of bread be between 800 g minus 5%, which is 760 g, and plus 10%, which is 880 g. Since every one of the measurements made by Vishnu lies within this range and since the means and medians are all close to 800 g, we can conclude that the supplier is reliable.

DEFINITION: Outlier

An outlier is a value in the data set that is not typical of the rest of the set. It is usually a value that is much greater or much less than all the other values in the data set.

Example 8: *Effect of outliers on mean and median*

QUESTION

The heights of 10 learners are measured in centimetres to obtain the following data set:

$$\{150;\ 172;\ 153;\ 156;\ 146;\ 157;\ 157;\ 143;\ 168;\ 157\}$$

Afterwards, we include one more learner in the group, who is exceptionally tall at 181 cm. Compare the mean and median of the heights of the learners before and after the 11^{th} learner was included.

SOLUTION

Step 1 : **Calculate the mean of the first 10 learners**

$$\begin{aligned}\text{Mean} &= \frac{150+172+153+156+146+157+157+143+168+157}{10} \\ &= 155{,}9 \text{ cm}\end{aligned}$$

Step 2 : **Calculate the mean of all 11 learners**

$$\text{Mean} = \frac{150 + 172 + 153 + 156 + 146 + 157 + 157 + 143 + 168 + 157 + 181}{11}$$
$$= 158{,}2 \text{ cm}$$

From this we see that the average height changes by $158{,}2 - 155{,}9 = 2{,}3$ cm when we introduce the outlier value (the tall person) to the data set.

Step 3 : ***Calculate the median of the first** 10 **learners***

To find the median, we need to sort the data set:

$$\{143;\ 146;\ 150;\ 153;\ 156;\ 157;\ 157;\ 157;\ 168;\ 172\}$$

Since there are an even number of values, 10, the median lies halfway between the 5^{th} and 6^{th} values:

$$\text{Median} = \frac{156 + 157}{2} = 156{,}5 \text{ cm}$$

Step 4 : ***Calculate the median of all** 11 **learners***

After adding the tall learner, the sorted data set is

$$\{143;\ 146;\ 150;\ 153;\ 156;\ 157;\ 157;\ 157;\ 168;\ 172;\ 181\}$$

Now, with 11 values, the median is the 6^{th} value: 157 cm. So, the median changes by only $0{,}5$ cm when we add the outlier value to the data set. In general, the median is less affected by the addition of outliers to a data set than the mean is. This is important because it is quite common that outliers are measured during an experiment, because of problems with the equipment or unexpected interference.

Exercise 9 - 1

1. Calculate the mean, median and mode of the following data sets:

(a) $2;\ 5;\ 8;\ 8;\ 11;\ 13;\ 22;\ 23;\ 27$

(b) $15;\ 17;\ 24;\ 24;\ 26;\ 28;\ 31;\ 43$

(c) $4;\ 11;\ 3;\ 15;\ 11;\ 13;\ 25;\ 17;\ 2;\ 11$

(d) $24;\ 35;\ 28;\ 41;\ 32;\ 49;\ 31$

2. The ages of 15 runners of the Comrades Marathon were recorded:

$$31;\ 42;\ 28;\ 38;\ 45;\ 51;\ 33;\ 29;\ 42;\ 26;\ 34;\ 56;\ 33;\ 46;\ 41$$

Calculate the mean, median and modal age.

3. In the first of a series of jars, there is 1 sweet. In the second jar, there are 3 sweets. The mean number of sweets in the first two jars is 2.
   (a) If the mean number of sweets in the first three jars is 3, how many sweets are there in the third jar?
   (b) If the mean number of sweets in the first four jars is 4, how many sweets are there in the fourth jar?
4. Find a set of five ages for which the mean age is 5, the modal age is 2 and the median age is 3 years.
5. Four friends each have some marbles. They work out that the mean number of marbles they have is 10. One friend leaves with 4 marbles. How many marbles do the remaining friends have together?

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(1.) 00jr (2.) 00js (3.) 00jt (4.) 00ju (5.) 00jv

9.3 Grouping data EMADI

A common way of handling continuous quantitative data is to subdivide the full range of values into a few sub-ranges. By assigning each continuous value to the sub-range or class within which it falls, the data set changes from continuous to discrete.

Grouping is done by defining a set of ranges and then counting how many of the data fall inside each range. The sub-ranges must not overlap and must cover the entire range of the data set.

One way of visualising grouped data is as a histogram. A histogram is a collection of rectangles, where the base of a rectangle (on the x-axis) covers the values in the range associated with it, and the height of a rectangle corresponds to the number of values in its range.

Video: VMbue at www.everythingmaths.co.za

Example 9: *Groups and histograms*

QUESTION

The heights in centimetres of 30 *learners are given below.*

142	163	169	132	139	140	152	168	139	150
161	132	162	172	146	152	150	132	157	133
141	170	156	155	169	138	142	160	164	168

Group the data into the following ranges and draw a histogram of the grouped data:

$$130 \leq h < 140$$
$$140 \leq h < 150$$
$$150 \leq h < 160$$
$$160 \leq h < 170$$
$$170 \leq h < 180$$

(Note that the ranges do not overlap since each one starts where the previous one ended.)

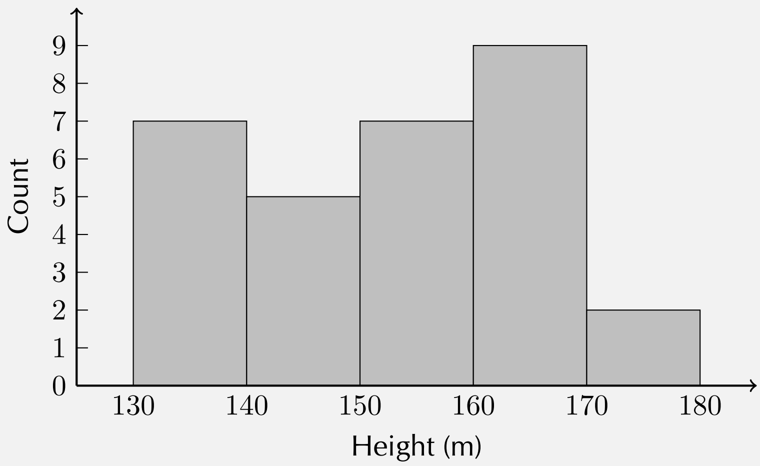
SOLUTION

Step 1 : ***Count the number of values in each range***

Range	Count
$130 \leq h < 140$	7
$140 \leq h < 150$	5
$150 \leq h < 160$	7
$160 \leq h < 170$	9
$170 \leq h < 180$	2

Step 2 : ***Draw the histogram***

Since there are 5 ranges, the histogram will have 5 rectangles. The base of each rectangle is defined by its range. The height of each rectangle is determined by the count in its range.



The histogram makes it easy to see in which range most of the heights are located and provides an overview of the distribution of the values in the data set.

Exercise 9 - 2

1. A class experiment was conducted and 50 learners were asked to guess the number of sweets in a jar. The following guesses were recorded:

56	49	40	11	33	33	37	29	30	59
21	16	38	44	38	52	22	24	30	34
42	15	48	33	51	44	33	17	19	44
47	23	27	47	13	25	53	57	28	23
36	35	40	23	45	39	32	58	22	40

(a) Draw up a grouped frequency table using the intervals $10 < x \leq 20$; $20 < x \leq 30$; $30 < x \leq 40$; $40 < x \leq 50$; and $50 < x \leq 60$.

(b) Draw the histogram corresponding to the frequency table of the grouped data.

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(1.) 00jw

Measures of central tendency EMADJ

With grouped data our estimates of central tendency will change because we lose some information when we place each value in a range. If all we have to work with is the grouped data, we do not know the measured values to the same accuracy as before. The best we can do is to assume that values are grouped at the centre of each range.

Looking back to the previous worked example, we started with this data set of learners' heights.

$$\{132;\ 132;\ 132;\ 133;\ 138;\ 139;\ 139;\ 140;\ 141;\ 142;\ 142;\ 146;\ 150;\ 150;\ 152;\ 152;\ 155;\ 156;\ 157;\ 160;\ 161;\ 162;\ 163;\ 164;\ 168;\ 168;\ 169;\ 169;\ 170;\ 172\}$$

Note that the data are sorted. The mean of these data is $151{,}8$ and the median is 152. The mode is 132, but remember that there are problems with computing the mode of continuous quantitative data.

After grouping the data, we now have the data set shown below. Note that each value is placed at the centre of its range and that the number of times that each value is repeated corresponds exactly to the counts in each range.

$$\{135;\ 135;\ 135;\ 135;\ 135;\ 135;\ 135;\ 145;\ 145;\ 145;\ 145;\ 145;\ 155;\ 155;\ 155;\ 155;\ 155;\ 155;\ 155;\ 155;\ 165;\ 165;\ 165;\ 165;\ 165;\ 165;\ 165;\ 165;\ 165;\ 175;\ 175\}$$

The grouping changes the measures of central tendency since each datum is treated as if it occurred at the centre of the range in which it was placed.

The mean is now 153, the median 155 and the mode is 165. This is actually a better estimate of the mode, since the grouping showed in which range the learners' heights were clustered.

Exercise 9 - 3

1. Consider the following grouped data and calculate the mean, the modal group and the median group.

Mass (kg)	**Count**
$40 < m \leq 45$	7
$45 < m \leq 50$	10
$50 < m \leq 55$	15
$55 < m \leq 60$	12
$60 < m \leq 65$	6

2. Find the mean, the modal group and the median group in this data set of how much time people needed to complete a game.

Time (s)	Count
$35 < t \leq 45$	5
$45 < t \leq 55$	11
$55 < t \leq 65$	15
$65 < t \leq 75$	26
$75 < t \leq 85$	19
$85 < t \leq 95$	13
$95 < t \leq 105$	6

3. The histogram below shows the number of passengers that travel in Alfred's minibus taxi per week.
Calculate

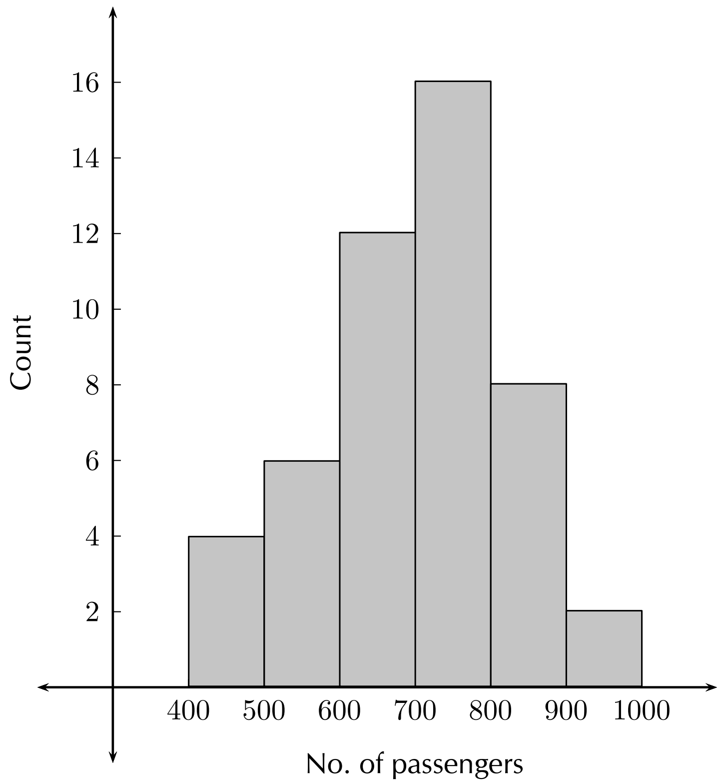
(a) the modal interval

(b) the total number of passengers to travel in Alfred's taxi

(c) an estimate of the mean

(d) an estimate of the median

(e) if it is estimated that every passenger travelled an average distance of 5 km, how much money would Alfred have made if he charged R $3{,}50$ per km?



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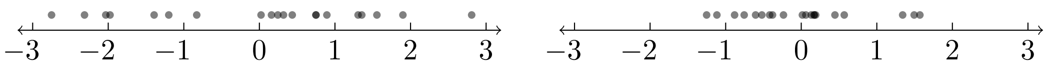
(1.) 00jx (2.) 00jy (3.) 00jz

9.4 Measures of dispersion EMADK

The central tendency is not the only interesting or useful information about a data set. The two data sets illustrated below have the same mean (0), but have different spreads around the mean. Each circle represents one datum.



Dispersion is a general term for different statistics that describe how values are distributed around the centre. In this section we will look at measures of dispersion.

Video: VMbsf at www.everythingmaths.co.za

Range EMADL

DEFINITION: Range

The range of a data set is the difference between the maximum and minimum values in the set.

The most straightforward measure of dispersion is the range. The range simply tells us how far apart the largest and smallest values in a data set are. The range is very sensitive to outliers.

Example 10: *Range*

QUESTION

Find the range of the following data set:

$$\{1;\ 4;\ 5;\ 8;\ 6;\ 7;\ 5;\ 6;\ 7;\ 4;\ 10;\ 9;\ 10\}$$

What would happen if we removed the first datum from the set?

SOLUTION

Step 1 : ***Determine the range***

The smallest value in the data set is 1 and the largest value is 10. The range is $10 - 1 = 9$.

Step 2 : ***Remove the first datum***

If the first datum, 1, were to be removed from the set, the minimum value would be 4. This means that the range would change to $10-4 = 6$. This is very different from the previous value. This shows that the range is sensitive to outliers. In the data set above, the value 1 is not typical of the other values. It is an outlier and has a big influence on the range.

Percentiles

EMADM

DEFINITION: Percentile

The p^{th} percentile is the value, v, that divides a data set into two parts, such that p percent of the values in the data set are less than v and $100-p$ percent of the values are greater than v. Percentiles can lie in the range $0 \leq p \leq 100$.

To understand percentiles properly, we need to distinguish between 3 different aspects of a datum: its value, its rank and its percentile:

- The value of a datum is what we measured and recorded during an experiment or survey.
- The rank of a datum is its position in the sorted data set (for example, first, second, third, and so on).
- The percentile at which a particular datum is, tells us what percentage of the values in the full data set are less than this datum.

The table below summarises the value, rank and percentile of the data set

$$\{14{,}2;\ 13{,}9;\ 19{,}8;\ 10{,}3;\ 13{,}0;\ 11{,}1\}$$

Value	**Rank**	**Percentile**
10,3	1	0
11,1	2	20
13,0	3	40
13,9	4	60
14,2	5	80
19,8	6	100

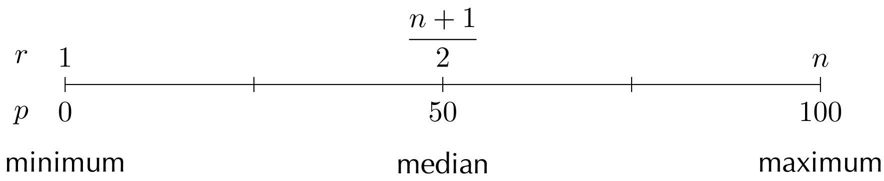
As an example, $13{,}0$ is at the 40^{th} percentile since there are 2 values less than $13{,}0$ and 3 values greater than $13{,}0$.

$$\frac{2}{2+3} = 0{,}4 = 40\%$$

In general, the formula for finding the p^{th} percentile in an ordered data set with n values is

$$r = \frac{p}{100}(n-1)+1$$

This gives us the rank, r, of the p^{th} percentile. To find the value of the p^{th} percentile, we have to count from the first value in the ordered data set up to the r^{th} value. Sometimes the rank will not be an integer. This means that the percentile lies between two values in the data set. The convention is to take the value halfway between the two values indicated by the rank. The figure below shows the relationship between rank and percentile graphically. We have already encountered three percentiles in this chapter: the median (50^{th} percentile), the minimum (0^{th} percentile) and the maximum (100^{th}). The median is defined as the value halfway in a sorted data set.



Example 11: *Using the percentile formula*

QUESTION

Determine the minimum, maximum and median values of the following data set using the percentile formula.

$$\{14;\ 17;\ 45;\ 20;\ 19;\ 36;\ 7;\ 30;\ 8\}$$

SOLUTION

Step 1 : ***Sort the values in the data set***

Before we can use the rank to find values in the data set, we always have to order the values from the smallest to the greatest. The sorted data set is

$$\{7;\ 8;\ 14;\ 17;\ 19;\ 20;\ 30;\ 36;\ 45\}$$

Step 2 : ***Find the minimum***

We already know that the minimum value is the first value in the ordered data set. We will now confirm that the percentile formula gives the same answer. The minimum is equivalent to the 0^{th} percentile. According to the percentile formula the rank, r, of the $p = 0^{th}$ percentile in a data set with $n = 9$ values is

$$\begin{aligned} r &= \frac{p}{100}(n-1)+1 \\ &= \frac{0}{100}(9-1)+1 \\ &= 1 \end{aligned}$$

This confirms that the minimum value is the first value in the list, namely 7.

Step 3 : ***Find the maximum***

We already know that the maximum value is the last value in the ordered data set. The maximum is also equivalent to the 100^{th} percentile. Using the percentile formula with $p = 100$ and $n = 9$, we find the rank of the maximum value as

$$\begin{aligned} r &= \frac{p}{100}(n-1)+1 \\ &= \frac{100}{100}(9-1)+1 \\ &= 9 \end{aligned}$$

This confirms that the maximum value is the last (the 9^{th}) value in the list, namely 45.

Step 4 : ***Find the median***

The median is equivalent to the 50^{th} percentile. Using the percentile formula with $p = 50$ and $n = 9$, we find the rank of the median value as

$$\begin{aligned} r &= \frac{50}{100}(n-1)+1 \\ &= \frac{50}{100}(9-1)+1 \\ &= \frac{1}{2}(8)+1 \\ &= 5 \end{aligned}$$

This shows that the median is in the middle (at the 5^{th} position) of the ordered data set. Therefore the median value is 19.

DEFINITION: Quartiles

The quartiles are the three data values that divide an ordered data set into four groups, where each group contains an equal number of data values. The median (50^{th} percentile) is the second quartile ($Q2$). The 25^{th} percentile is also be called the first or lower quartile ($Q1$), and the 75^{th} percentile the third or upper quartile ($Q3$).

Example 12: *Quartiles*

QUESTION

Determine the quartiles of the following data set:

$$\{7;\ 45;\ 11;\ 3;\ 9;\ 35;\ 31;\ 7;\ 16;\ 40;\ 12;\ 6\}$$

SOLUTION

Step 1 : ***Sort the data set***

$$\{3;\ 6;\ 7;\ 7;\ 9;\ 11;\ 12;\ 16;\ 31;\ 35;\ 40;\ 45\}$$

Step 2 : ***Find the ranks of the quartiles***

Using the percentile formula, $n = 12$, we can find the rank of the

25^{th}, 50^{th} and 75^{th} percentiles as

$$\begin{aligned} r_{25} &= \frac{25}{100}(12-1)+1 \\ &= 3{,}75 \\ r_{50} &= \frac{50}{100}(12-1)+1 \\ &= 6{,}5 \\ r_{75} &= \frac{75}{100}(12-1)+1 \\ &= 9{,}25 \end{aligned}$$

Step 3 : ***Find the values of the quartiles***

Note that each of these ranks is a fraction, meaning that the value for each percentile is somewhere in between two values from the data set. For the 25^{th} percentile the rank is $3{,}75$, which is between the 3^{rd} and 4^{th} values. Since both these values are equal to 7, the 25^{th} percentile is 7. For the 50^{th} percentile (the median) the rank is $6{,}5$, meaning halfway between the 6^{th} and 7^{th} values. The 6^{th} value is 11 and the 7^{th} value is 12, which means that the median is $\frac{11+12}{2} = 11{,}5$. For the 75^{th} percentile the rank is $9{,}25$, meaning between the 9^{th} and 10^{th} values. Therefore the 75^{th} percentile is $\frac{31+35}{2} = 33$.

Deciles

The deciles are the nine data values that divide an ordered data set into ten groups, where each group contains an equal number of data values.

For example, consider the ordered data set:

$$\begin{gathered} 28; 33; 35; 45; 57; 59; 61; 68; 69; 72; 75; 78; 80; 83; 86; 91; \\ 92; 95; 101; 105; 111; 117; 118; 125; 127; 131; 137; 139; 141 \end{gathered}$$

The nine deciles are: $35; 59; 69; 78; 86; 95; 111; 125; 137$

Percentiles for grouped data

EMADN

In grouped data, the percentiles will lie somewhere inside a range, rather than at a specific value. To find the range in which a percentile lies, we still use the percentile formula to determine the rank of the percentile and then find the range within which that rank is.

Example 13: Percentiles in grouped data

QUESTION

The mathematics marks of 100 grade 10 learners at a school have been collected. The data are presented in the following table:

Percentage mark	**Number of learners**
$0 \leq x < 20$	2
$20 \leq x < 30$	5
$30 \leq x < 40$	18
$40 \leq x < 50$	22
$50 \leq x < 60$	18
$60 \leq x < 70$	13
$70 \leq x < 80$	12
$80 \leq x < 100$	10

1. *Calculate the mean of this grouped data set.*
2. *In which intervals are the quartiles of the data set?*
3. *In which interval is the 30^{th} percentile of the data set?*

SOLUTION

Step 1 : ***Calculate the mean***

Since we are given grouped data rather than the original ungrouped data, the best we can do is approximate the mean as if all the learners in each interval were located at the central value of the interval.

$$\text{Mean} = \frac{2\,.\,10 + 5\,.\,25 + 18\,.\,35 + 22\,.\,45 + 18\,.\,55 + 13\,.\,65 + 12\,.\,75 + 10\,.\,90}{100}$$
$$= 54\%$$

Step 2 : ***Find the quartiles***

Since the data have been grouped, they have also already been sorted. Using the percentile formula and the fact that there are 100 learners, we can find the rank of the 25^{th}, 50^{th} and 75^{th} percentiles as

$$\begin{aligned} r_{25} &= \frac{25}{100}(100-1)+1 \\ &= 24{,}75 \\ r_{50} &= \frac{50}{100}(100-1)+1 \\ &= 50{,}5 \\ r_{75} &= \frac{75}{100}(100-1)+1 \\ &= 75{,}25 \end{aligned}$$

Now we need to find in which ranges each of these ranks lie.

- For the lower quartile, we have that there are $2+5=7$ learners in the first two ranges combined and $2+5+18=25$ learners in the first three ranges combined. Since $7 < r_{25} < 25$, this means that the lower quartile lies somewhere in the third range: $30 \leq x < 40$.
- For the second quartile (the median), we have that there are $2+5+18+22 = 47$ learners in the first four ranges combined and 65 learners in the first five ranges combined. Since $47 < r_{50} < 65$, this means that the median lies somewhere in the fifth range: $50 \leq x < 60$.
- For the upper quartile, we have that there are 65 learners in the first five ranges combined and $65+13=78$ learners in the first six ranges combined. Since $65 < r_{75} < 78$, this means that the upper quartile lies somewhere in the sixth range: $60 \leq x < 70$.

Step 3 : ***Find the 30^{th} percentile***

Using the same method as for the quartiles, we first find the rank of the 30^{th} percentile.

$$\begin{aligned} r &= \frac{30}{100}(100-1)+1 \\ &= 30{,}7 \end{aligned}$$

Now we have to find the range in which this rank lies. Since there are 25 learners in the first 3 ranges combined and 47 learners in the first 4 ranges combined, the 30^{th} percentile lies in the fourth range: $40 \leq x < 50$.

Ranges

EMADO

We define data ranges in terms of percentiles. We have already encountered the full data range, which is simply the difference between the 100^{th} and the 0^{th} percentile (that is, between the maximum and minimum values in the data set).

DEFINITION: Interquartile range

The interquartile range is a measure of dispersion, which is calculated by subtracting the first quartile ($Q1$) from the third quartile ($Q3$). This gives the range of the middle half of the data set.

DEFINITION: Semi interquartile range

The semi interquartile range is half of the interquartile range.

Exercise 9 - 4

1. Find the range of the data set

$$\{1;\ 2;\ 3;\ 4;\ 4;\ 4;\ 5;\ 6;\ 7;\ 8;\ 8;\ 9;\ 10;\ 10\}$$

2. What are the quartiles of this data set?

$$\{3;\ 5;\ 1;\ 8;\ 9;\ 12;\ 25;\ 28;\ 24;\ 30;\ 41;\ 50\}$$

3. A class of 12 students writes a test and the results are as follows:

$$20;\ 39;\ 40;\ 43;\ 43;\ 46;\ 53;\ 58;\ 63;\ 70;\ 75;\ 91$$

   Find the range, quartiles and the interquartile range.

4. Three sets of data are given:
   - **Data set 1:** $\{9;\ 12;\ 12;\ 14;\ 16;\ 22;\ 24\}$
   - **Data set 2:** $\{7;\ 7;\ 8;\ 11;\ 13;\ 15;\ 16;\ 16\}$
   - **Data set 3:** $\{11;\ 15;\ 16;\ 17;\ 19;\ 19;\ 22;\ 24;\ 27\}$

   For each data set find:

   (a) the range
   (b) the lower quartile
   (c) the interquartile range
   (d) the semi-interquartile range
   (e) the median
   (f) the upper quartile

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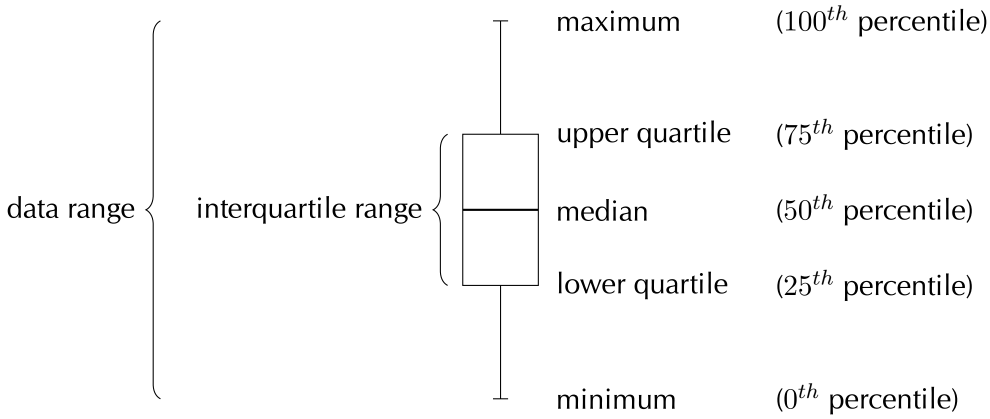
(1.) 00k0 (2.) 00k1 (3.) 00k2 (4.) 00k3

9.5 Five number summary EMADP

A common way of summarising the overall data set is with the five number summary and the box-and-whisker plot. These two represent exactly the same information, numerically in the case of the five number summary and graphically in the case of the box-and-whisker plot.

The five number summary consists of the minimum value, the maximum value and the three quartiles. Another way of saying this is that the five number summary consists of the following percentiles: 0^{th}, 25^{th}, 50^{th}, 75^{th}, 100^{th}. The box-and-whisker plot shows these five percentiles as in the figure below. The box shows the interquartile range (the distance between $Q1$ and $Q3$). A line inside the box shows the median. The lines extending outside the box (the whiskers) show where the minimum and maximum values lie. This graph can also be drawn horizontally.

Video: VMbur at www.everythingmaths.co.za



Example 14: *Five number summary*

QUESTION

Draw a box and whisker diagram for the following data set:

$$\{1{,}25;\ 1{,}5;\ 2{,}5;\ 2{,}5;\ 3{,}1;\ 3{,}2;\ 4{,}1;\ 4{,}25;\ 4{,}75;\ 4{,}8;\ 4{,}95;\ 5{,}1\}$$

SOLUTION

Step 1 : ***Determine the minimum and maximum***

Since the data set is already sorted, we can read off the minimum as the first value $(1{,}25)$ and the maximum as the last value $(5{,}1)$.

Step 2 : ***Determine the quartiles***

There are 12 values in the data set. Using the percentile formula, we can determine that the median lies between the 6^{th} and 7^{th} values, making it

$$\frac{3{,}2 + 4{,}1}{2} = 3{,}65$$

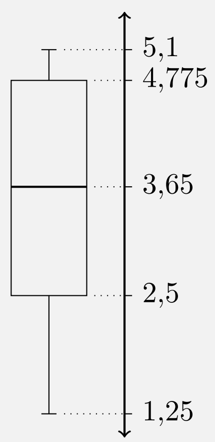
The first quartile lies between the 3^{rd} and 4^{th} values, making it

$$\frac{2{,}5 + 2{,}5}{2} = 2{,}5$$

The third quartile lies between the 9^{th} and 10^{th} values, making it

$$\frac{4{,}75 + 4{,}8}{2} = 4{,}775$$

This provides the five number summary of the data set and allows us to draw the following box-and-whisker plot.



Exercise 9 - 5

1. Lisa is working in a computer store. She sells the following number of computers each month:

$$\{27;\ 39;\ 3;\ 15;\ 43;\ 27;\ 19;\ 54;\ 65;\ 23;\ 45;\ 16\}$$

Give the five number summary and box-and-whisker plot of Lisa's sales.

2. Zithulele works as a telesales person. He keeps a record of the number of sales he makes each month. The data below show how much he sells each month.

$$\{49;\ 12;\ 22;\ 35;\ 2;\ 45;\ 60;\ 48;\ 19;\ 1;\ 43;\ 12\}$$

Give the five number summary and box-and-whisker plot of Zithulele's sales.

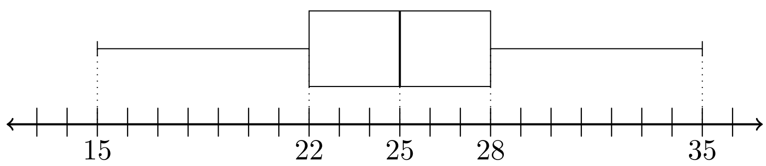
3. Hannah has worked as a florist for nine months. She sold the following number of wedding bouquets:

$$\{16;\ 14;\ 8;\ 12;\ 6;\ 5;\ 3;\ 5;\ 7\}$$

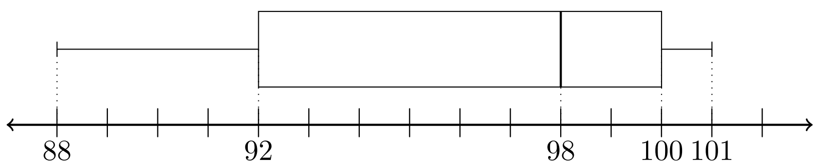
Give the five number summary of Hannah's sales.

4. Use the diagram below to determine the five number summary:

(a)



(b)



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(1.) 00k4 (2.) 00k5 (3.) 00k6 (4.) 00k7

Chapter 9 | Summary

Summary presentation: VMdvy at www.everythingmaths.co.za

- Data refer to the pieces of information that have been observed and recorded, from an experiment or a survey.
- Quantitative data are data that can be written as numbers. Quantitative data can be discrete or continuous.
- Qualitative data are data that cannot be written as numbers. There are two common types of qualitative data: categorical and anecdotal data.
- The mean is the sum of a set of values divided by the number of values in the set.

$$\begin{aligned}\overline{x} &= \frac{1}{n}\sum_{i=1}^{n} x_i \\ &= \frac{x_1 + x_2 + \cdots + x_n}{n}\end{aligned}$$

- The median of a data set is the value in the central position, when the data set has been arranged from the lowest to the highest value. If there are an odd number of data, the median will be equal to one of the values in the data set. If there are an even number of data, the median will lie half way between two values in the data set.
- The mode of a data set is the value that occurs most often in the set.
- An outlier is a value in the data set that is not typical of the rest of the set. It is usually a value that is much greater or much less than all the other values in the data set.
- Dispersion is a general term for different statistics that describe how values are distributed around the centre.
- The range of a data set is the difference between the maximum and minimum values in the set.
- The p^{th} percentile is the value, v, that divides a data set into two parts, such that $p\%$ of the values in the data set are less than v and $100 - p\%$ of the values are greater than v. The general formula for finding the p^{th} percentile in an ordered data set with n values is

$$r = \frac{p}{100}(n-1) + 1$$

- The quartiles are the three data values that divide an ordered data set into four groups, where each group contains an equal number of data values. The lower quartile is denoted $Q1$, the median is $Q2$ and the upper quartile is $Q3$.
- The interquartile range is a measure of dispersion, which is calculated by subtracting the lower (first) quartile from the upper (third) quartile. This gives the range of the middle half of the data set.
- The semi interquartile range is half of the interquartile range.
- The five number summary consists of the minimum value, the maximum value and the three quartiles ($Q1$, $Q2$ and $Q3$).
- The box-and-whisker plot is a graphical representation of the five number summary.

Chapter 9 End of Chapter Exercises

1. In a park, the tallest 7 trees have heights (in metres):

$$41;\ 60;\ 47;\ 42;\ 44;\ 42;\ 47$$

Find the median of their heights.

2. The students in Ndeme's class have the following ages:

$$5;\ 6;\ 7;\ 5;\ 4;\ 6;\ 6;\ 6;\ 7;\ 4$$

Find the mode of their ages.

3. An engineering company has designed two different types of engines for motorbikes. The two different motorbikes are tested for the time (in seconds) it takes for them to accelerate from 0 km/h to 60 km/h.

	Test 1	**Test 2**	**Test 3**	**Test 4**	**Test 5**	**Test 6**	**Test 7**	**Test 8**	**Test 9**	**Test 10**
Bike 1	1,55	1,00	0,92	0,80	1,49	0,71	1,06	0,68	0,87	1,09
Bike 2	0,9	1,0	1,1	1,0	1,0	0,9	0,9	1,0	0,9	1,1

(a) Which measure of central tendency should be used for this information?

(b) Calculate the measure of central tendency that you chose in the previous question, for each motorbike.

(c) Which motorbike would you choose based on this information? Take note of the accuracy of the numbers from each set of tests.

4. In a traffic survey, a random sample of 50 motorists were asked the distance they drove to work daily. This information is shown in the table below.

Distance (km)	Count
$0 < d \leq 5$	4
$5 < d \leq 10$	5
$10 < d \leq 15$	9
$15 < d \leq 20$	10
$20 < d \leq 25$	7
$25 < d \leq 30$	8
$30 < d \leq 35$	3
$35 < d \leq 40$	2
$40 < d \leq 45$	2

(a) Find the approximate mean of the data.

(b) What percentage of samples had a distance of

i. less than 15 km?

ii. more than 30 km?

iii. between 16 km and 30 km daily?

(c) Draw a histogram to represent the data

5. A company wanted to evaluate the training programme in its factory. They gave the same task to trained and untrained employees and timed each one in seconds.

Trained	121	137	131	135	130
	128	130	126	132	127
	129	120	118	125	134
Untrained	135	142	126	148	145
	156	152	153	149	145
	144	134	139	140	142

(a) Find the medians and quartiles for both sets of data.
(b) Find the interquartile range for both sets of data.
(c) Comment on the results.
(d) Draw a box-and-whisker diagram for each data set to illustrate the five number summary.

6. A small firm employs 9 people. The annual salaries of the employers are:

R 600 000	R 250 000	R 200 000
R 120 000	R 100 000	R 100 000
R 100 000	R 90 000	R 80 000

(a) Find the mean of these salaries.
(b) Find the mode.
(c) Find the median.
(d) Of these three figures, which would you use for negotiating salary increases if you were a trade union official? Why?

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(1.) 00k8 (2.) 00k9 (3.) 00ka (4.) 00kb (5.) 00kc

(6.) 00kd

Probability 10

We use probability to describe uncertain events. When you accidentally drop a slice of bread, you don't know if it's going to fall with the buttered side facing upwards or downwards. When your favourite sports team plays a game, you don't know whether they will win or not. When the weatherman says that there is a 40% chance of rain tomorrow, you may or may not end up getting wet. Uncertainty presents itself to some degree in every event that occurs around us and in every decision that we make.

We will see in this chapter that all of these uncertainties can be described using the rules of probability theory and that we can make definite conclusions about uncertain processes.

We'll use three examples of uncertain processes to help you understand the meanings of the different words used in probability theory: tossing a coin, rolling dice, and a soccer match.

Video: VMbxk at www.everythingmaths.co.za

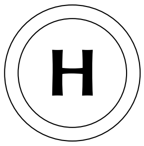
DEFINITION: Experiment

An experiment refers to an uncertain process.

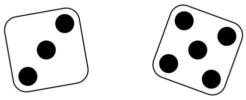
DEFINITION: Outcome

An outcome of an experiment is a single result of that experiment.

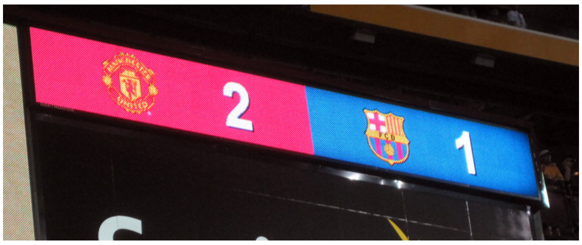
Experiment 1 A coin is tossed and it lands with either heads (H) or tails (T) facing upwards. An example outcome of tossing a coin is that it lands with heads facing up:



Experiment 2 Two dice are rolled and the total number of dots added up. An example outcome of rolling two dice:



Experiment 3 Two teams play a soccer match and we are interested in the final score. An example outcome of a soccer match:



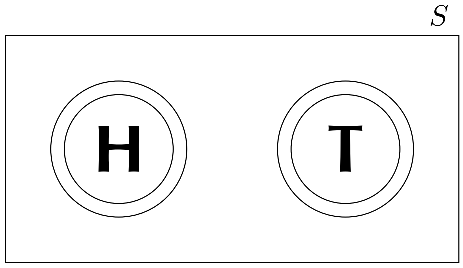
(Picture by apasciuto on Flickr.com)

DEFINITION: Sample space

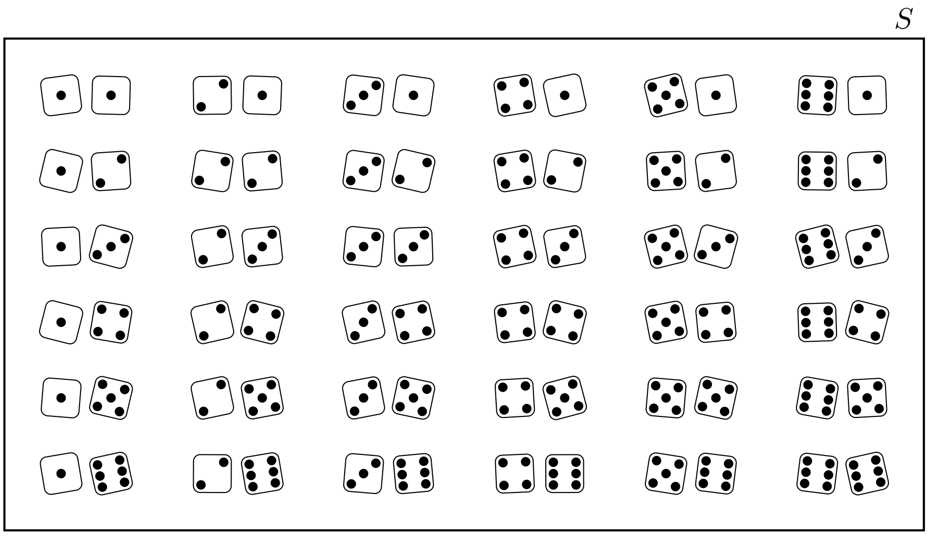
The sample space of an experiment is the set of all possible outcomes of that experiment. The sample space is denoted with the symbol S and the size of the sample space (the total number of possible outcomes) is denoted with $n(S)$.

Even though we are usually interested in the outcome of an experiment, we also need to know what the other outcomes could have been. Let's have a look at the sample spaces of each of our three experiments.

Experiment 1 Since a coin can land in one of only two ways (we will ignore the possibility that the coin lands on its edge), the sample space is the set $S = \{\text{H}; \text{T}\}$. The size of the sample space of the coin toss is $n(S) = 2$:



Experiment 2 Each of the dice can land on a number from 1 to 6. In this experiment the sample space of all possible outcomes is every possible combination of the 6 numbers on the first die with the 6 numbers on the second die. This gives a total of $n(S) = 6 \times 6 = 36$ possible outcomes. The figure below shows all of the outcomes in the sample space of rolling two dice:



Experiment 3 Each soccer team can get an integer score from 0 upwards. Usually we don't expect a score to go much higher than 5 goals, but there is no reason why this cannot happen. So the sample space of this experiment consists of all possible combinations of two non-negative integers. The figure below shows all of the possibilities. Since we do not limit the score of a team, this sample space is infinitely large:

S

0 – 0	1 – 0	2 – 0	3 – 0	$\cdots$
0 – 1	1 – 1	2 – 1	3 – 1	$\cdots$
0 – 2	1 – 2	2 – 2	3 – 2	$\cdots$
0 – 3	1 – 3	2 – 3	3 – 3	
$\vdots$	$\vdots$	$\vdots$		$\ddots$

DEFINITION: Event

An event is a specific set of outcomes of an experiment that you are interested in. An event is denoted with the letter E and the number of outcomes in the event with $n(E)$.

Experiment 1 Let's say that we would like the coin to land heads up. Here the event contains a single outcome: $E = \{\text{H}\}$, $n(E) = 1$.

Experiment 2 Let's say that we are interested in the sum of the dice being 8. In this case the event set is $E = \{$(⚁ ⚅); (⚂ ⚄); (⚃ ⚃); (⚄ ⚂); (⚅ ⚁)$\}$ since it contains all of the possible ways to get 8 dots with 2 dice. The size of the event set is $n(E) = 5$.

Experiment 3 We would like to know whether the first team will win. For this event to happen the first score must be greater than the second.

$$E = \{(1;0);(2;0);(2;1);(3;0);(3;1);(3;2);\ldots\}$$

This event set is infinitely large.

10.1 Theoretical probability

EMADQ

DEFINITION: Probability

A probability is a real number between 0 and 1 that describes how likely it is that an event will occur.

It can also be described as a percentage (a probability of $0{,}75$ can be written as 75%) or as fraction ($0{,}75$ can also be written as $\frac{3}{4}$).

- A probability of 0 means that an event will never occur.
- A probability of 1 means that an event will always occur.
- A probability of $0{,}5$ means that an event will occur half the time, or 1 time out of every 2.

When all of the possible outcomes of an experiment have an equal chance of occurring, we can compute the exact theoretical probability of an event. The probability of an event is the ratio between the number of outcomes in the event set and the number of possible outcomes in the sample space.

$$P(E) = \frac{n(E)}{n(S)}$$

Video: VMbzp at www.everythingmaths.co.za

Example 1: *Theoretical probabilities*

QUESTION

What is the theoretical probability of each of the events in the first two of our three experiments?

SOLUTION

Step 1 : ***Write down the value of*** $n(S)$

Experiment 1 (coin): $n(S) = 2$
Experiment 2 (dice): $n(S) = 36$

Step 2 : ***Write down the size of the event set***
Experiment 1: $n(E) = 1$
Experiment 2: $n(E) = 5$

Step 3 : ***Compute the theoretical probability***
Experiment 1: $P(E) = \frac{n(E)}{n(S)} = \frac{1}{2} = 0{,}5$

Experiment 2: $P(E) = \frac{n(E)}{n(S)} = \frac{5}{36} = 0{,}13\dot{8}$

Note that we do not consider the theoretical probability of the third experiment. The third experiment is different from the first two in an important way, namely that all possible outcomes (all final scores) are not equally likely. For example, we know that a soccer score of 1–1 is quite common, while a score of 11–15 is very, very rare. Because all outcomes are not equally likely, we cannot use the ratio between $n(E)$ and $n(S)$ to compute the theoretical probability of a team winning.

Exercise 10 - 1

1. A bag contains 6 red, 3 blue, 2 green and 1 white balls. A ball is picked at random. Determine the probability that it is:
   (a) red
   (b) blue or white
   (c) not green
   (d) not green or red
2. A playing card is selected randomly from a pack of 52 cards. Determine the probability that it is:
   (a) the 2 of hearts
   (b) a red card
   (c) a picture card
   (d) an ace

(e) a number less than 4

3. Even numbers from 2 to 100 are written on cards. What is the probability of selecting a multiple of 5, if a card is drawn at random?

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(1.) 00id (2.) 00ie (3.) 00if

10.2 Relative frequency EMADR

DEFINITION: Relative frequency

The relative frequency of an event is defined as the number of times that the event occurs during experimental trials, divided by the total number of trials conducted.

The relative frequency is not a theoretical quantity, but an experimental one. We have to repeat an experiment a number of times and count how many times the outcome of the trial is in the event set. Because it is experimental, it is possible to get a different relative frequency every time that we repeat an experiment.

Video: VMbzy at www.everythingmaths.co.za

Example 2: *Relative frequency and theoretical probability*

QUESTION

We toss a coin 30 times and observe the outcomes. The results of the trials are shown in the table below.

trial	1	2	3	4	5	6	7	8	9	10
outcome	*H*	*T*	*T*	*T*	*H*	*T*	*H*	*H*	*H*	*T*
trial	11	12	13	14	15	16	17	18	19	20
outcome	*H*	*T*	*T*	*H*	*T*	*T*	*T*	*H*	*T*	*T*
trial	21	22	23	24	25	26	27	28	29	30
outcome	*H*	*H*	*H*	*T*	*H*	*T*	*H*	*T*	*T*	*T*

What is the relative frequency of observing heads after each trial and how does it compare to the theoretical probability of observing heads?

SOLUTION

Step 1 : ***Count the number of positive outcomes***

A positive outcome is when the outcome is in our event set. The table below shows a running count (after each trial t) of the number of positive outcomes p we have observed. For example, after $t = 20$ trials we have observed heads 8 times and tails 12 times and so the positive outcome count is $p = 8$.

t	1	2	3	4	5	6	7	8	9	10
p	1	1	1	1	2	2	3	4	5	5
t	11	12	13	14	15	16	17	18	19	20
p	6	6	6	7	7	7	7	8	8	8
t	21	22	23	24	25	26	27	28	29	30
p	9	10	11	11	12	12	13	13	13	13

Step 2 : ***Compute the relative frequency***

Since the relative frequency is defined as the ratio between the number of positive trials and the total number of trials,

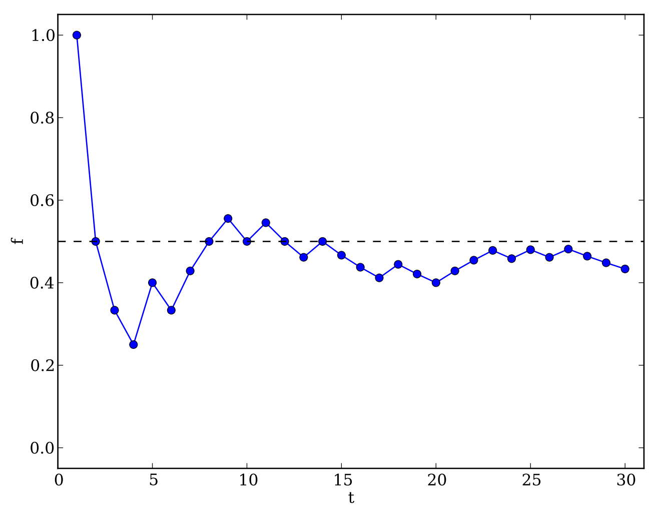
$$f = \frac{p}{t}$$

The relative frequency of observing heads, f, after having completed t coin tosses is:

t	1	2	3	4	5	6	7	8	9	10
f	1,00	0,50	0,33	0,25	0,40	0,33	0,43	0,50	0,56	0,50
t	11	12	13	14	15	16	17	18	19	20
f	0,55	0,50	0,46	0,50	0,47	0,44	0,41	0,44	0,42	0,40
t	21	22	23	24	25	26	27	28	29	30
f	0,43	0,45	0,48	0,46	0,48	0,46	0,48	0,46	0,45	0,43

From the last entry in this table we can now easily read the relative frequency after 30 trials, namely $13/30 = 0{,}4\dot{3}$. The relative frequency is close to the theoretical probability of $0{,}5$. In general, the relative frequency of an event tends to get closer to the theoretical probability of the event as we perform more trials.

A much better way to summarise the table of relative frequencies is in a graph:



The graph above is the plot of the relative frequency of observing heads, f, after having completed t coin tosses. It was generated from the table of numbers above by plotting the number of trials that have been completed, t, on the x-axis and the relative frequency, f, on the y-axis. In the beginning (after a small number of trials) the relative frequency fluctuates a lot around the theoretical probability at 0,5, which is shown with a dashed line. As the number of trials increases, the relative frequency fluctuates less and gets closer to the theoretical probability.

Example 3: *Relative frequency and theoretical probability*

QUESTION

While watching 10 soccer games where Team 1 plays against Team 2, we record the following final scores:

Trial	1	2	3	4	5	6	7	8	9	10
Team 1	2	0	1	1	1	1	1	0	5	3
Team 2	0	2	2	2	2	1	1	0	0	0

What is the relative frequency of Team 1 winning?

SOLUTION

In this experiment, each trial takes the form of Team 1 playing a soccer match against Team 2.

Step 1 : ***Count the number of positive outcomes***

We are interested in the event where Team 1 wins. From the table above we see that this happens 3 times.

Step 2 : ***Compute the relative frequency***

The total number of trials is 10. This means that the relative frequency of the event is

$$\frac{3}{10} = 0{,}3$$

It is important to understand the difference between the theoretical probability of an event and the observed relative frequency of the event in experimental trials. The theoretical probability is a number that we can compute if we have enough information about the experiment. If each possible outcome in the sample space is equally likely, we can count the number of outcomes in the event set and the number of outcomes in the sample space to compute the theoretical probability.

The relative frequency depends on the sequence of outcomes that we observe while doing a statistical experiment. The relative frequency can be different every time we redo the experiment. The more trials we run during an experiment, the closer the observed relative frequency of an event will get to the theoretical probability of the event.

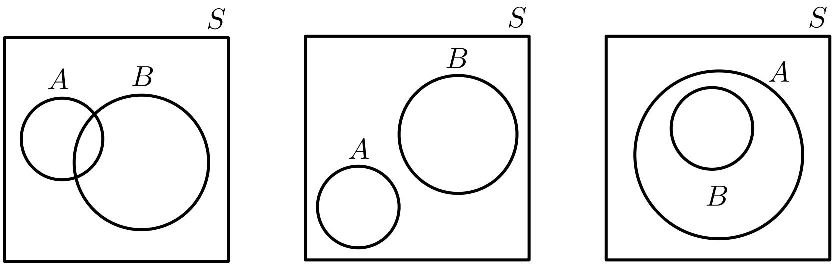
So why do we need statistical experiments if we have theoretical probabilities? In some cases, like our soccer experiment, it is difficult or impossible to compute the theoretical probability of an event. Since we do not know exactly how likely it is that one soccer team will score goals against another, we can never compute the theoretical probability of events in soccer. In such cases we can still use the relative frequency to estimate the theoretical probability, by running experiments and counting the number of positive outcomes.

10.3 Venn diagrams

EMADS

A Venn diagram is a graphical way of representing the relationships between sets. In each Venn diagram a set is represented by a closed curve. The region inside the curve represents the elements that belong to the set, while the region outside the curve represents the elements that are excluded from the set.

Venn diagrams are helpful for thinking about probability since we deal with different sets. Consider two events, A and B, in a sample space S. The diagram below shows the possible ways in which the event sets can overlap, represented using Venn diagrams:



The sets are represented using a rectangle for S and circles for each of A and B. In the first diagram the two events overlap partially. In the second diagram the two events do not overlap at all. In the third diagram one event is fully contained in the other. Note that events will always appear inside the sample space since the sample space contains all possible outcomes of the experiment.

Video: VMcbf at www.everythingmaths.co.za

Example 4: *Venn diagrams*

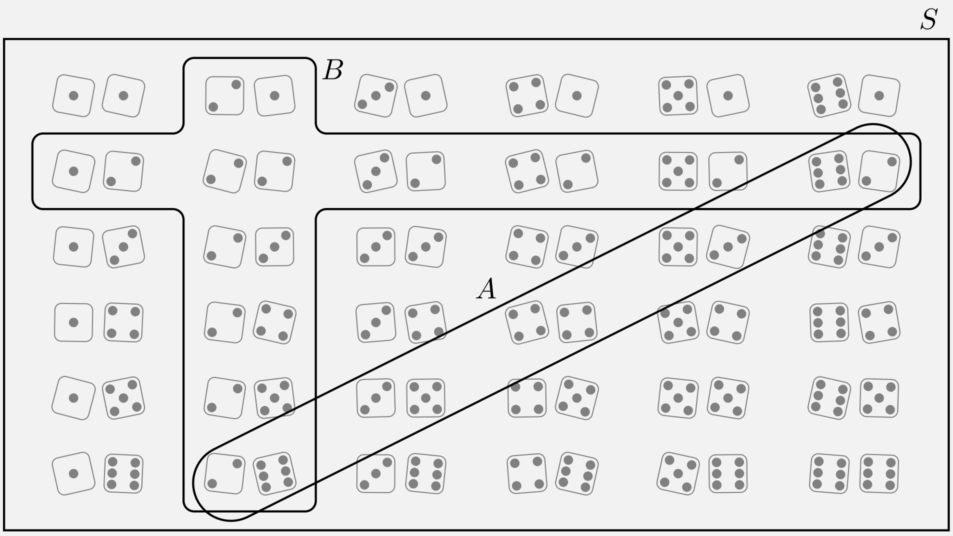
QUESTION

Represent the sample space of two rolled dice and the following two events using a Venn diagram:

Event A: the sum of the dice equals 8

Event B: at least one of the dice shows

SOLUTION



Example 5: *Venn diagrams*

QUESTION

Consider the set of diamonds removed from a deck of cards. A random card is selected from the set of diamonds.

- *Write down the sample space, S, for the experiment.*
- *What is the value of $n(S)$?*
- *Consider the following two events:*
  - P*: An even diamond is chosen*
  - R*: A royal diamond is chosen*

  Represent sample space S and events P and R using a Venn diagram.

SOLUTION

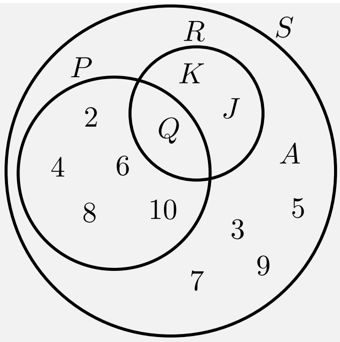
Step 1 : ***Write down the sample space*** S

$$S = \{A;\ 2;\ 3;\ 4;\ 5;\ 6;\ 7;\ 8;\ 9;\ 10;\ J;\ Q;\ K\}$$

Step 2 : ***Write down the value of*** $n(S)$

$$n(S) = 13$$

Step 3 : ***Draw the Venn diagram***



Exercise 10 - 2

1. Let S denote the set of whole numbers from 1 to 16, X denote the set of even numbers from 1 to 16 and Y denote the set of prime numbers from 1 to 16. Draw a Venn diagram depicting S, X and Y.

2. There are 79 Grade 10 learners at school. All of these take some combination of Maths, Geography and History. The number who take Geography is 41, those who take History is 36, and 30 take Maths. The number who take Maths and History is 16; the number who take Geography and History is 6, and there are 8 who take Maths only and 16 who take History only.
   (a) Draw a Venn diagram to illustrate all this information.
   (b) How many learners take Maths and Geography but not History?
   (c) How many learners take Geography only?
   (d) How many learners take all three subjects?

3. Pieces of paper labelled with the numbers 1 to 12 are placed in a box and the box is shaken. One piece of paper is taken out and then replaced.
   (a) What is the sample space, S?
   (b) Write down the set A, representing the event of taking a piece of paper labelled with a factor of 12.
   (c) Write down the set B, representing the event of taking a piece of paper labelled with a prime number.
   (d) Represent A, B and S by means of a Venn diagram.
   (e) Find
      i. $n(S)$

ii. $n(A)$
iii. $n(B)$

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(1.) 00ig (2.) 00ih (3.) 00ii

10.4 Union and intersection

EMADT

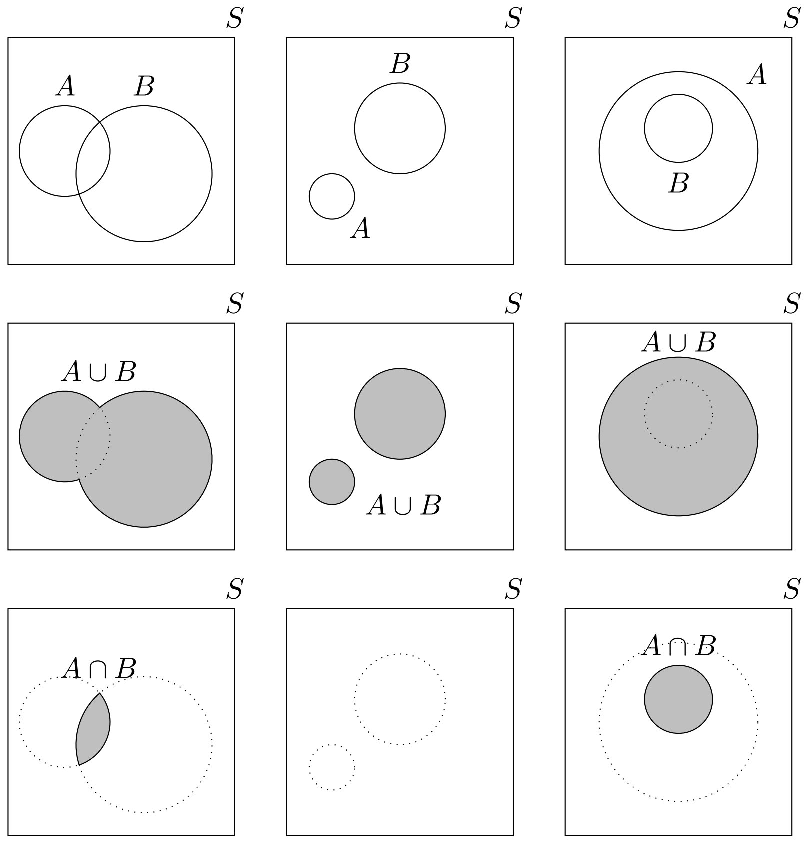
DEFINITION: Union

The union of two sets is a new set that contains all of the elements that are in at least one of the two sets. The union is written as $A \cup B$.

DEFINITION: Intersection

The intersection of two sets is a new set that contains all of the elements that are in both sets. The intersection is written as $A \cap B$.

The figure below shows the union and intersection for different configurations of two events in a sample space, using Venn diagrams.



The unions and intersections of different events. Note that in the middle column the intersection, $A \cap B$, is empty since the two sets do not overlap. In the final column the union, $A \cup B$, is equal to A and the intersection, $A \cap B$, is equal to B since B is fully contained in A.

Video: VMcci at www.everythingmaths.co.za

10.5 Probability identities

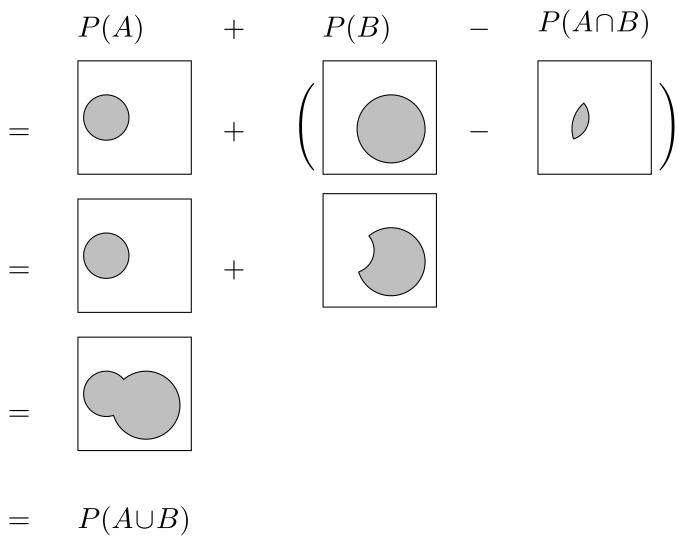
EMADU

$$P(S) = 1$$

By definition, the sample space contains all possible outcomes of an experiment. So we know that the probability of observing an outcome from the sample space is 1.

$$P(A \cup B) = P(A) + P(B) - P(A \cap B)$$

We will prove this identity using the Venn diagrams given above. For each of the 4 terms in the union and intersection identity, we can draw the Venn diagram and then add and subtract the different diagrams. The area of a region represents its probability. We will do this for the first column of the Venn diagram figure given previously. You should also try it for the other columns.



Example 6: *Union and intersection of events*

QUESTION

Relate the probabilities of events A and B from Example 4 (two rolled dice) and show that they satisfy the identity

$$P(A \cup B) = P(A) + P(B) - P(A \cap B).$$

SOLUTION

Step 1 : ***Write down the probabilities of the two events, their union and their intersection***

From the Venn diagram in Example 4, we can count the number of outcomes in each event. To get the probability of an event, we divide the size of the event by the size of the sample space, which is

$n(S) = 36$.

$$\begin{aligned} P(A) &= \frac{n(A)}{n(S)} &= \frac{5}{36} \\ P(B) &= \frac{n(B)}{n(S)} &= \frac{11}{36} \\ P(A \cap B) &= \frac{n(A \cap B)}{n(S)} &= \frac{2}{36} \\ P(A \cup B) &= \frac{n(A \cup B)}{n(S)} &= \frac{14}{36} \end{aligned}$$

Step 2 : ***Write down and check the identity***

$$\begin{aligned} P(A \cup B) &= P(A) + P(B) - P(A \cap B) \\ \frac{14}{36} &= \frac{5}{36} + \frac{11}{36} - \frac{2}{36} \\ &= \frac{5}{36} + \frac{9}{36} \\ &= \frac{14}{36} \end{aligned}$$

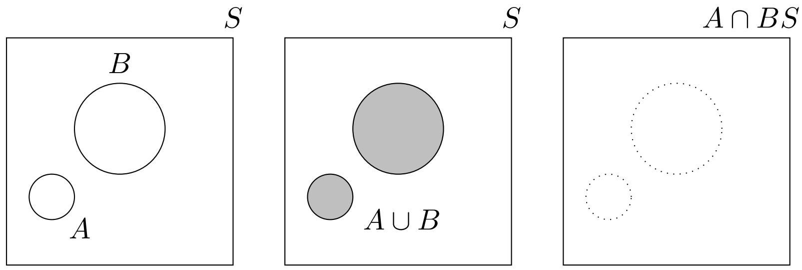
10.6 Mutually exclusive events

EMADV

DEFINITION: Mutually exclusive events

Two events are called mutually exclusive if they cannot occur at the same time. Whenever an outcome of an experiment is in the first event it can not also be in the second event, and vice versa.

Another way of saying this is that the two event sets, A and B, cannot have any elements in common, or $P(A \cap B) = \emptyset$ (where $\emptyset$ denotes the empty set). We have already seen the Venn diagrams of mutually exclusive events in the middle column of the Venn diagrams provided on page 336.



From this figure you can see that the intersection has no elements. You can also see that the probability of the union is the sum of the probabilities of the events.

$$P(A \cup B) = P(A) + P(B)$$

This relationship is true for mutually exclusive events only.

Example 7: *Mutually exclusive events*

QUESTION

We roll two dice and are interested in the following two events:

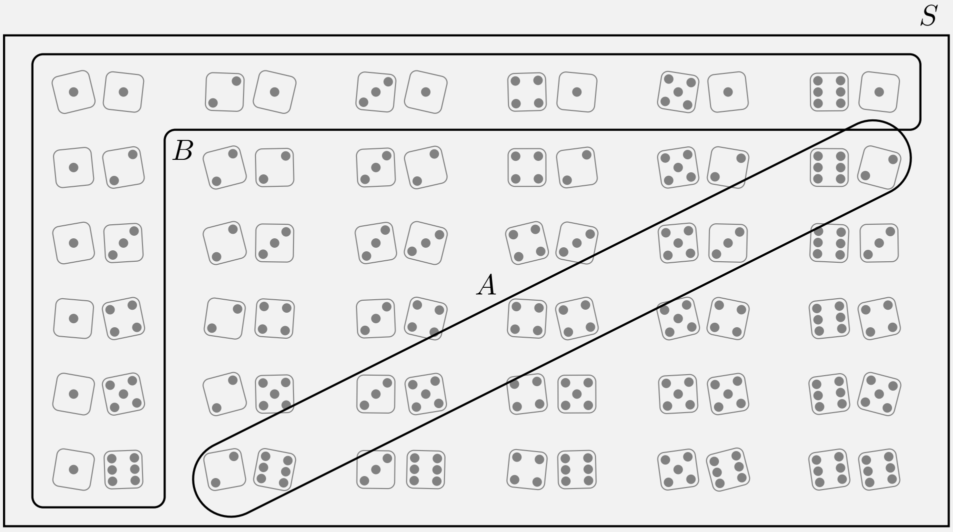
A : *The sum of the dice equals* 8

B : *At least one of the dice shows* ⚀

Show that the events are mutually exclusive.

SOLUTION

Step 1 : **Draw the sample space and the two events**



Step 2 : ***Determine the intersection***

From the above figure we notice that there are no elements in common in A and B. Therefore the events are mutually exclusive.

10.7 Complementary events

EMADW

DEFINITION: Complementary set

The complement of a set, A, is a different set that contains all of the elements that are not in A. We write the complement of A as A', or sometimes as "not (A)".

For an experiment with sample space S and an event A we can derive some identities for complementary events. Since every element in A is not in A', we know that complementary events are mutually exclusive.

$$A \cap A' = \emptyset$$

Since every element in the sample space is either in A or in A', the union of complementary events covers the sample space.

$$A \cup A' = S$$

From the previous two identities, we also know that the probabilities of complementary events sum to 1.

$$P(A) + P(A') = P(A \cup A') = P(S) = 1$$

Video: VMcdl at www.everythingmaths.co.za

Example 8: *Reasoning with Venn diagrams*

QUESTION

In a survey 70 people were questioned about which product they use: A or B or both. The report of the survey shows that 25 people use product A, 35 people use product B and 15 people use neither. Use a Venn diagram to work out how many people

1. *use product A only*
2. *use product B only*
3. *use both product A and product B*

SOLUTION

Step 1 : ***Summarise the sizes of the sample space, the event sets, their union and their intersection***

- We are told that 70 people were questioned, so the size of the sample space is $n(S) = 70$.
- We are told that 25 people use product A, so $n(A) = 25$.
- We are told that 35 people use product B, so $n(B) = 35$.
- We are told that 15 people use neither product. This means that $70-15 = 55$ people use at least one of the two products, so $n(A \cup B) = 55$.

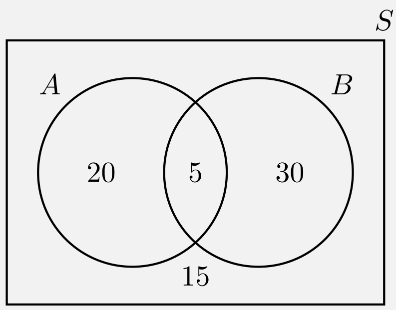
- We are not told how many people use both products, so we have to work out the size of the intersection, $A \cap B$, by using the identity

$$
\begin{aligned}
P(A \cup B) &= P(A) + P(B) - P(A \cap B) \\
\frac{n(A \cup B)}{n(S)} &= \frac{n(A)}{n(S)} + \frac{n(B)}{n(S)} - \frac{n(A \cap B)}{n(S)} \\
\frac{55}{70} &= \frac{25}{70} + \frac{35}{70} - \frac{n(A \cap B)}{70} \\
\therefore n(A \cap B) &= 25 + 35 - 55 \\
&= 5
\end{aligned}
$$

Step 2 : ***Determine whether the events are mutually exclusive***

Since the intersection of the events, $A \cap B$, is not empty, the events are not mutually exclusive. This means that their circles should overlap in the Venn diagram.

Step 3 : ***Draw the Venn diagram and fill in the numbers***



Step 4 : ***Read off the answers***

1. 20 people use product A only.
2. 30 people use product B only.
3. 5 people use both products.

Exercise 10 - 3

1. A box contains coloured blocks. The number of each colour is given in the following table.

Colour	Purple	Orange	White	Pink
Number of blocks	24	32	41	19

A block is selected randomly. What is the probability that the block will be:

   (a) purple
   (b) purple or white
   (c) pink and orange
   (d) not orange?

2. A small school has a class with children of various ages. The table gives the number of pupils of each age in the class.

	3 years old	4 years old	5 years old
Male	2	7	6
Female	6	5	4

If a pupil is selected at random what is the probability that the pupil will be:

   (a) a female
   (b) a 4 year old male
   (c) aged 3 or 4
   (d) aged 3 and 4
   (e) not 5
   (f) either 3 or female?

3. Fiona has 85 labelled discs, which are numbered from 1 to 85. If a disc is selected at random what is the probability that the disc number:

   (a) ends with 5
   (b) is a multiple of 3
   (c) is a multiple of 6
   (d) is number 65
   (e) is not a multiple of 5
   (f) is a multiple of 4 or 3

(g) is a multiple of 2 and 6
(h) is number 1?

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(1.) 00in (2.) 00ip (3.) 00iq

Chapter 10 | Summary

Summary presentation: VMdve at www.everythingmaths.co.za

- An experiment refers to an uncertain process.
- An outcome of an experiment is a single result of that experiment.
- The sample space of an experiment is the set of all possible outcomes of that experiment. The sample space is denoted with the symbol S and the size of the sample space (the total number of possible outcomes) is denoted with $n(S)$.
- An event is a specific set of outcomes of an experiment that you are interested in. An event is denoted with the letter E and the number of outcomes in the event with $n(E)$.
- A probability is a real number between 0 and 1 that describes how likely it is that an event will occur.
  - A probability of 0 means that an event will never occur.
  - A probability of 1 means that an event will always occur.
  - A probability of $0{,}5$ means that an event will occur half the time, or 1 time out of every 2.
- A probability can also be written as a percentage or as a fraction.
- The relative frequency of an event is defined as the number of times that the event occurs during experimental trials, divided by the total number of trials conducted.
- The union of two sets is a new set that contains all of the elements that are in at least one of the two sets. The union is written as $A \cup B$.
- The intersection of two sets is a new set that contains all of the elements that are in both sets. The intersection is written as $A \cap B$.

- The probability of sample space: $P(S) = 1$.
- Union and intersection: $P(A \cup B) = P(A) + P(B) - P(A \cap B)$.
- Mutually exclusive events are two events that cannot occur at the same time. Whenever an outcome of an experiment is in the first event, it can not also be in the second event.
- The complement of a set, A, is a different set that contains all of the elements that are not in A. We write the complement of A as A', or sometimes as "not(A)".
- Complementary events are mutually exclusive: $A \cap A' = \emptyset$.
- Complementary events cover the sample space: $A \cup A' = S$.
- Probabilities of complementary events sum to 1: $P(A) + P(A') = P(A \cup A') = P(S) = 1$.

Chapter 10 End of Chapter Exercises

1. A group of 45 children were asked if they eat Frosties and/or Strawberry Pops. 31 eat both and 6 eat only Frosties. What is the probability that a child chosen at random will eat only Strawberry Pops?
2. In a group of 42 pupils, all but 3 had a packet of chips or a Fanta or both. If 23 had a packet of chips and 7 of these also had a Fanta, what is the probability that one pupil chosen at random has:
   (a) both chips and Fanta
   (b) only Fanta
3. Use a Venn diagram to work out the following probabilities for a die being rolled:
   (a) a multiple of 5 and an odd number
   (b) a number that is neither a multiple of 5 nor an odd number
   (c) a number which is not a multiple of 5, but is odd
4. A packet has yellow and pink sweets. The probability of taking out a pink sweet is $\frac{7}{12}$. What is the probability of taking out a yellow sweet?
5. In a car park with 300 cars, there are 190 Opels. What is the probability that the first car to leave the car park is:
   (a) an Opel
   (b) not an Opel

6. Tamara has 18 loose socks in a drawer. Eight of these are orange and two are pink. Calculate the probability that the first sock taken out at random is:
   - **(a)** orange
   - **(b)** not orange
   - **(c)** pink
   - **(d)** not pink
   - **(e)** orange or pink
   - **(f)** neither orange nor pink
7. A plate contains 9 shortbread cookies, 4 ginger biscuits, 11 chocolate chip cookies and 18 Jambos. If a biscuit is selected at random, what is the probability that:
   - **(a)** it is either a ginger biscuit of a Jambo
   - **(b)** it is not a shortbread cookie
8. 280 tickets were sold at a raffle. Ingrid bought 15 tickets. What is the probability that Ingrid:
   - **(a)** wins the prize
   - **(b)** does not win the prize
9. The children in a nursery school were classified by hair and eye colour. 44 had red hair and not brown eyes, 14 had brown eyes and red hair, 5 had brown eyes but not red hair and 40 did not have brown eyes or red hair.
   - **(a)** How many children were in the school?
   - **(b)** What is the probability that a child chosen at random has:
     - **i.** brown eyes
     - **ii.** red hair
   - **(c)** A child with brown eyes is chosen randomly. What is the probability that this child will have red hair?
10. A jar has purple, blue and black sweets in it. The probability that a sweet chosen at random will be purple is $\frac{1}{7}$ and the probability that it will be black is $\frac{3}{5}$.
    - **(a)** If I choose a sweet at random what is the probability that it will be:
      - **i.** purple or blue
      - **ii.** black
      - **iii.** purple
    - **(b)** If there are 70 sweets in the jar how many purple ones are there?
    - **(c)** $\frac{2}{5}$ of the purple sweets in (b) have streaks on them and the rest do not. How many purple sweets have streaks?

11. For each of the following, draw a Venn diagram to represent the situation and find an example to illustrate the situation.

(a) a sample space in which there are two events that are not mutually exclusive

(b) a sample space in which there are two events that are complementary

12. Use a Venn diagram to prove that the probability of either event A or B occurring (A and B are not mutually exclusive) is given by:

$$P(A \cup B) = P(A) + P(B) - P(A \cap B)$$

13. All the clubs are taken out of a pack of cards. The remaining cards are then shuffled and one card chosen. After being chosen, the card is replaced before the next card is chosen.

(a) What is the sample space?

(b) Find a set to represent the event, P, of drawing a picture card.

(c) Find a set for the event, N, of drawing a numbered card.

(d) Represent the above events in a Venn diagram.

(e) What description of the sets P and N is suitable? (Hint: Find any elements of P in N and of N in P).

14. A survey was conducted at Mutende Primary School to establish how many of the 650 learners buy vetkoek and how many buy sweets during break. The following was found:

- 50 learners bought nothing
- 400 learners bought vetkoek
- 300 learners bought sweets

(a) Represent this information with a Venn diagram

(b) If a learner is chosen randomly, calculate the probability that this learner buys:

i. sweets only

ii. vetkoek only

iii. neither vetkoek nor sweets

iv. vetkoek and sweets

v. vetkoek or sweets

15. In a survey at Lwandani's Secondary School 80 people were questioned to find out how many read the Sowetan and how many read the Daily Sun newspaper or both. The survey revealed that 45 read the Daily Sun, 30 read the Sowetan and 10 read neither. Use a Venn diagram to find the percentage of people that read:

(a) Only the Daily Sun
(b) Only the Sowetan
(c) Both the Daily Sun and the Sowetan

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(1.) 00ir (2a-b.) 00is (3a-c.) 023q (4.) 023r (5a-b.) 00mp (6a-f.) 023s

(7a-b.) 023t (8a-b.) 023u (9a-c.) 023v (10a-c.) 00mq (11a-b.) 023w (12.) 023x

(13a-e.) 00it (14a-c.) 023i (15a-c.) 02sn

Euclidean geometry 11

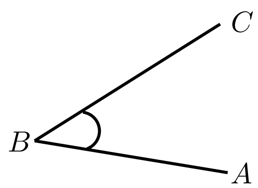
Geometry (from the Greek "geo" = earth and "metria" = measure) arose as the field of knowledge dealing with spatial relationships. Geometry can be split into Euclidean geometry and analytical geometry. Analytical geometry deals with space and shape using algebra and a coordinate system. Euclidean geometry deals with space and shape using a system of logical deductions.

Video: VMchw at www.everythingmaths.co.za

Angles

EMADX

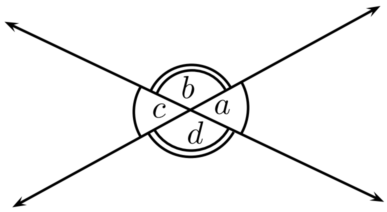
An angle is formed when two straight lines meet at a point, also known as a vertex. Angles are labelled with a caret on a letter, for example, $\hat{B}$. Angles can also be labelled according to the line segments that make up the angle, for example, $C\hat{B}A$ or $A\hat{B}C$. The $\angle$ symbol is a short method of writing angle in geometry and is often used for phrases such as "sum of $\angle$s in a $\triangle$". Angles are measured in degrees which is denoted by $^\circ$, a small circle raised above the text, similar to an exponent.



Properties and notation

EMADY

In the diagram below two straight lines intersect at a point, forming the four angles $\hat{a}$, $\hat{b}$, $\hat{c}$ and $\hat{d}$.



The following table summarises the different types of angles, with examples from the figure above.

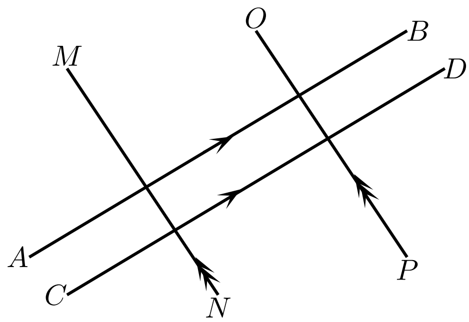
Term	Property	Examples
Acute angle	$0° <$ angle $< 90°$	$\hat{a}$; $\hat{c}$
Right angle	Angle $= 90°$	
Obtuse angle	$90° <$ angle $< 180°$	$\hat{b}$; $\hat{d}$
Straight angle	Angle $= 180°$	$\hat{a}+\hat{b}$; $\hat{b}+\hat{c}$
Reflex angle	$180° <$ angle $< 360°$	$\hat{a}+\hat{b}+\hat{c}$
Adjacent angles	Angles that share a vertex and a common side.	$\hat{a}$ and $\hat{d}$; $\hat{c}$ and $\hat{d}$
Vertically opposite angles	Angles opposite each other when two lines intersect. They share a vertex and are equal.	$\hat{a}=\hat{c}$; $\hat{b}=\hat{d}$
Supplementary angles	Two angles that add up to $180°$.	$\hat{a}+\hat{b}=180°$; $\hat{b}+\hat{c}=180°$
Complementary angles	Two angles that add up to $90°$.	
Revolution	The sum of all angles around a point.	$\hat{a}+\hat{b}+\hat{c}+\hat{d}=360°$

Note that adjacent angles on a straight line are supplementary.

Parallel lines and transversal lines

EMADZ

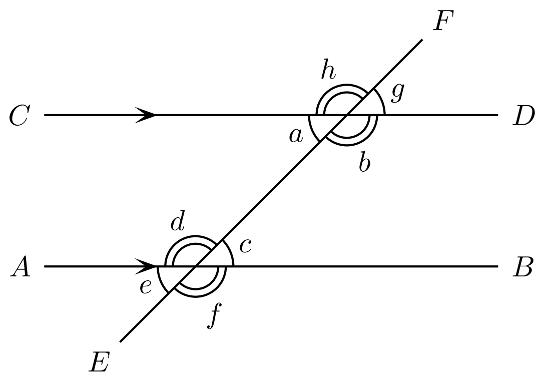
Two lines intersect if they cross each other at a point. For example, at a traffic intersection two or more streets intersect; the middle of the intersection is the common point between the streets. Parallel lines are always the same distance apart and they are denoted by arrow symbols as shown below.



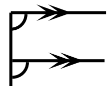
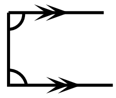
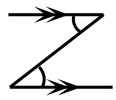
In writing we use two vertical lines to indicate that two lines are parallel:

$$AB \parallel CD \text{ and } MN \parallel OP$$

A transversal line intersects two or more parallel lines. In the diagram below, $AB \parallel CD$ and EF is a transversal line.



The properties of the angles formed by these intersecting lines are summarised in the following table:

Name of angle	**Definition**	**Examples**	**Notes**
Interior angles	Angles that lie in between the parallel lines.	$\hat{a}$, $\hat{b}$, $\hat{c}$ and $\hat{d}$ are interior angles.	Interior means inside.
Exterior angles	Angles that lie outside the parallel lines.	$\hat{e}$, $\hat{f}$, $\hat{g}$ and $\hat{h}$ are exterior angles.	Exterior means outside.
Corresponding angles	Angles on the same side of the lines and the same side of the transversal. If the lines are parallel, the corresponding angles will be equal.	$\hat{a}$ and $\hat{e}$, $\hat{b}$ and $\hat{f}$, $\hat{c}$ and $\hat{g}$, $\hat{d}$ and $\hat{h}$ are pairs of corresponding angles. $\hat{a} = \hat{e}$, $\hat{b} = \hat{f}$, $\hat{c} = \hat{g}$, $\hat{d} = \hat{h}$.	 F shape
Co-interior angles	Angles that lie in between the lines and on the same side of the transversal. If the lines are parallel, the angles are supplementary.	$\hat{a}$ and $\hat{d}$, $\hat{b}$ and $\hat{c}$ are pairs of co-interior angles. $\hat{a} + \hat{d} = 180°$, $\hat{b} + \hat{c} = 180°$.	 C shape
Alternate interior angles	Equal interior angles that lie inside the line and on opposite sides of the transversal. If the lines are parallel, the interior angles will be equal.	$\hat{a}$ and $\hat{c}$, $\hat{b}$ and $\hat{d}$ are pairs of alternate interior angles. $\hat{a} = \hat{c}$, $\hat{b} = \hat{d}$.	 Z shape

If two lines are intersected by a transversal such that

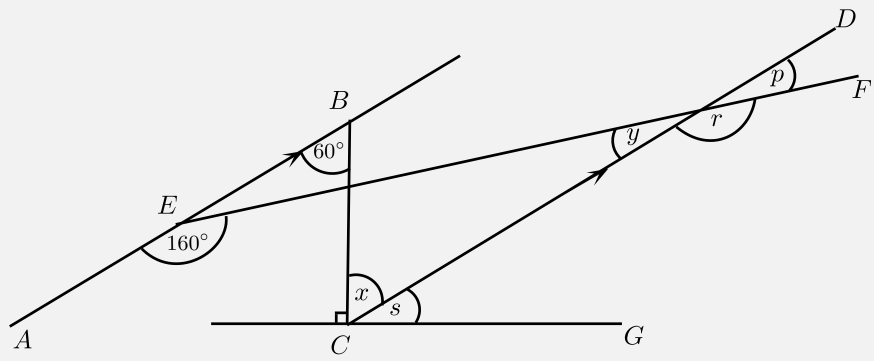
- corresponding angles are equal; or
- alternate interior angles are equal; or
- co-interior angles are supplementary

then the two lines are parallel.

Example 1: *Finding angles*

QUESTION

Find all the unknown angles. Is $EF \parallel CG$? Explain your answer.



SOLUTION

Step 1 : ***Use the properties of parallel lines to find all equal angles on the diagram***

Step 2 : ***Determine the unknown angles***

$$AB \parallel CD \quad \text{(given)}$$
$$\therefore \hat{x} = 60° \quad \text{(alt. int. } \angle\text{'s)}$$

$$\hat{y} + 160° = 180° \quad \text{(co-int. } \angle\text{'s)}$$
$$\therefore \hat{y} = 20°$$

$$\hat{p} = \hat{y} \quad \text{(vert. opp. } \angle\text{'s)}$$
$$\therefore \hat{p} = 20°$$

$$\hat{r} + \hat{p} = 180° \quad \text{(sum of } \angle\text{'s str. line)}$$
$$\therefore \hat{r} = 160°$$

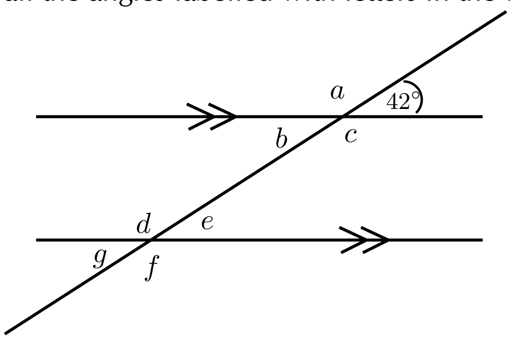
$$\hat{s} + \hat{x} = 90° \quad \text{(given)}$$
$$\hat{s} + 60° = 90°$$
$$\therefore \hat{s} = 30°$$

Step 3 : ***Determine whether*** $EF \parallel CG$

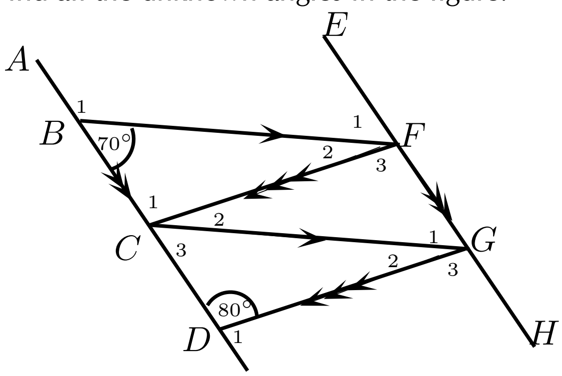
If $EF \parallel CG$ then $\hat{p}$ will be equal to corresponding angle $\hat{s}$, but $\hat{p} = 20°$ and $\hat{s} = 30°$. Therefore EF is not parallel to CG.

Exercise 11 - 1

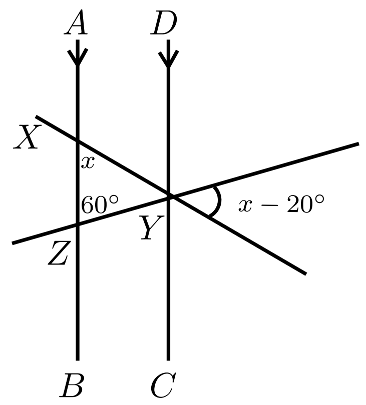
1. Use adjacent, corresponding, co-interior and alternate angles to fill in all the angles labelled with letters in the diagram:



2. Find all the unknown angles in the figure:

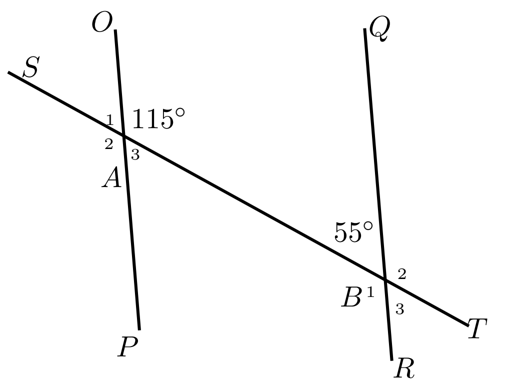


3. Find the value of x in the figure:

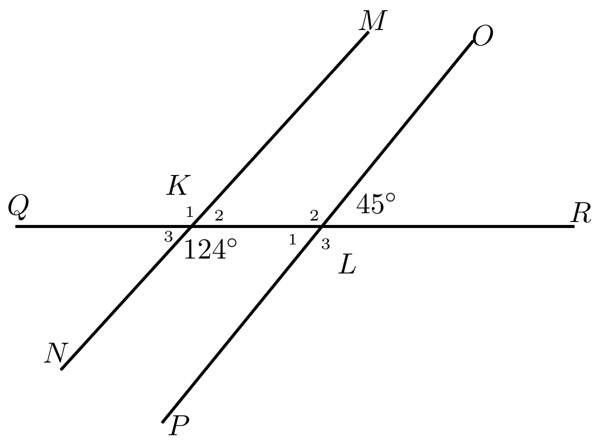


4. Determine whether the pairs of lines in the following figures are parallel:

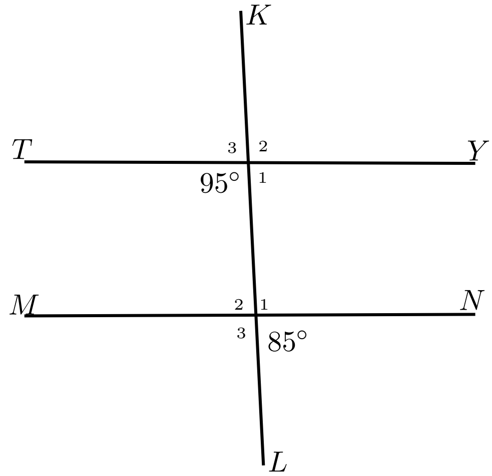
(a)



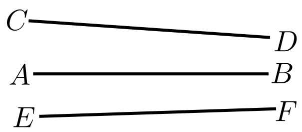
(b)



(c)



5. If AB is parallel to CD and AB is parallel to EF, explain why CD must be parallel to EF.



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(1.) 00gi (2.) 00gj (3.) 00gk (4.) 00gm (5.) 00gn

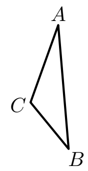
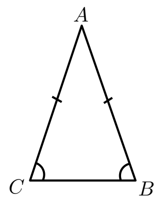
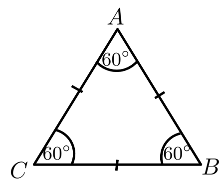
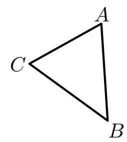
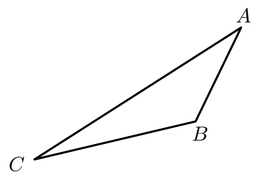
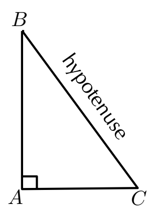
Classification of triangles

EMAEA

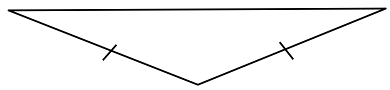
A triangle is a three-sided polygon. Triangles can be classified according to sides or according to angles: equilateral, isosceles and scalene, acute-angled, obtuse-angled and right-angled.

We use the notation $\triangle ABC$ to refer to a triangle with corners labelled A, B, and C.

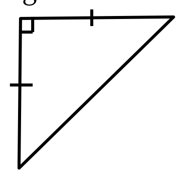
Video: VMciu at www.everythingmaths.co.za

Name	**Diagram**	**Properties**
Scalene		All sides and angles are different.
Isosceles		Two sides are equal in length. The angles opposite the equal sides are also equal.
Equilateral		All three sides are equal in length and all three angles are equal.
Acute		Each of the three interior angles is less than $90°$.
Obtuse		One interior angle is greater than $90°$.
Right-angled		One interior angle is $90°$.

Different combinations of these properties are also possible. For example, an obtuse isosceles triangle and a right-angled isosceles triangle are shown below:

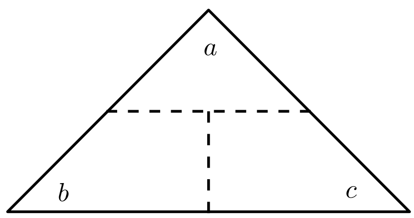


obtuse isosceles



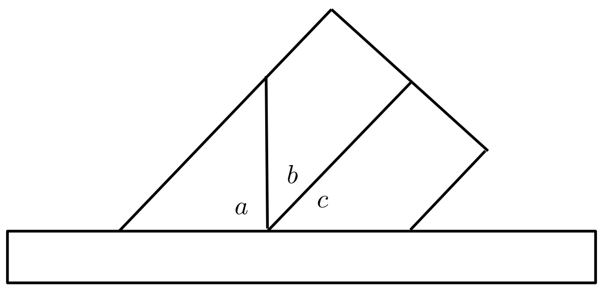
right-angled isosceles

Investigation:	*Interior angles of a triangle*

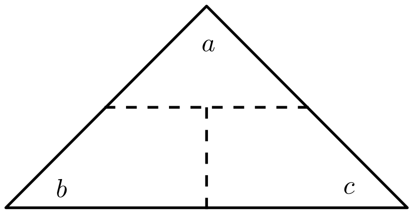


1. On a piece of paper draw a triangle of any size and shape.
2. Cut it out and label the angles $\hat{a}$, $\hat{b}$ and $\hat{c}$ on both sides of the paper.
3. Draw dotted lines as shown and cut along these lines to get three pieces of paper.
4. Place them along your ruler as shown in the figure below.
5. What can we conclude?

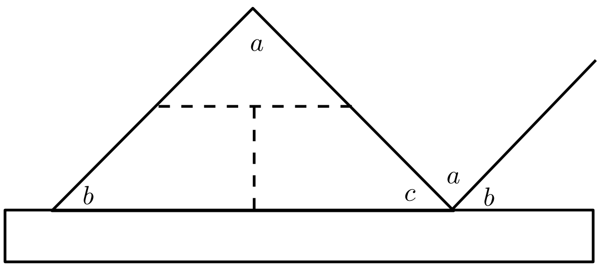
Hint: What is the sum of angles on a straight line?



Investigation:	*Exterior angles of a triangle*



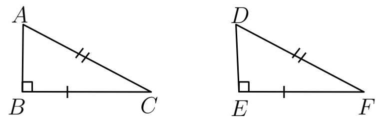
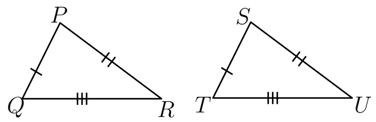
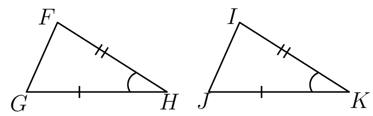
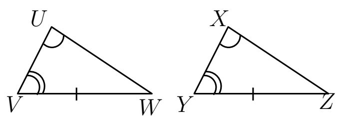
1. On a piece of paper draw a triangle of any size and shape. On another piece of paper, make a copy of the triangle.
2. Cut both out and label the angles of both triangles $\hat{a}$, $\hat{b}$ and $\hat{c}$ on both sides of the paper.
3. Draw dotted lines on **one** triangle as shown and cut along the lines.
4. Place the second triangle and the cut out pieces as shown in the figure below.
5. What can we can conclude?



Congruency

EMAAEB

Two triangles are congruent if one fits exactly over the other. This means that the triangles have equal corresponding angles and sides. To determine whether two triangles are congruent, it is not necessary to check every side and every angle. The following table describes the requirements for congruency:

Rule	**Description**	**Diagram**
RHS (90°, hypotenuse, side)	If the hypotenuse and one side of a right-angled triangle are equal to the hypotenuse and the corresponding side of another right-angled triangle, then the two triangles are congruent.	 $\triangle ABC \equiv \triangle DEF$
SSS (side, side, side)	If three sides of a triangle are equal in length to the corresponding sides of another triangle, then the two triangles are congruent.	 $\triangle PQR \equiv \triangle STU$
SAS (side, angle, side)	If two sides and the included angle of a triangle are equal to the corresponding two sides and **included** angle of another triangle, then the two triangles are congruent.	 $\triangle FGH \equiv \triangle IJK$
AAS (angle, angle, side)	If one side and two angles of a triangle are equal to the corresponding one side and two angles of another triangle, then the two triangles are congruent.	 $\triangle UVW \equiv \triangle XYZ$

The order of letters when labelling congruent triangles is very important.

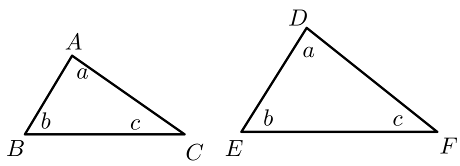
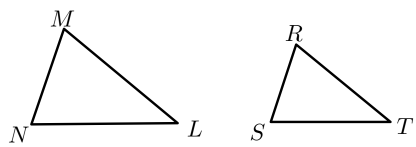
$$\triangle ABC \equiv \triangle DEF$$

This notation indicates the following properties of the two triangles: $\hat{A} = \hat{D}$, $\hat{B} = \hat{E}$, $\hat{C} = \hat{F}$, $AB = DE$, $AC = DF$ and $BC = EF$.

Similarity

EMAEC

Two triangles are similar if one triangle is a scaled version of the other. This means that their corresponding angles are equal in measure and the ratio of their corresponding sides are in proportion. The two triangles have the same shape, but different scales. Congruent triangles are similar triangles, but not all similar triangles are congruent. The following table describes the requirements for similarity:

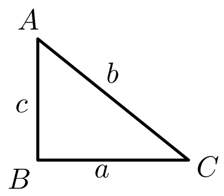
Rule	**Description**	**Diagram**
AAA (angle, angle, angle)	If all three pairs of corresponding angles of two triangles are equal, then the triangles are similar.	 $\hat{A} = \hat{D}$, $\hat{B} = \hat{E}$, $\hat{C} = \hat{F}$ $\therefore \triangle ABC \ \vert\vert\vert \ \triangle DEF$
SSS (side, side, side)	If all three pairs of corresponding sides of two triangles are in proportion, then the triangles are similar.	 $\frac{MN}{RS} = \frac{ML}{RT} = \frac{NL}{ST}$ $\therefore \triangle MNL \ \vert\vert\vert \ \triangle RST$

The order of letters for similar triangles is very important. Always label similar triangles in corresponding order. For example,

$\triangle MNL \ ||| \ \triangle RST$ is correct; but

$\triangle MNL \ ||| \ \triangle RTS$ is incorrect.

The theorem of Pythagoras EMAED

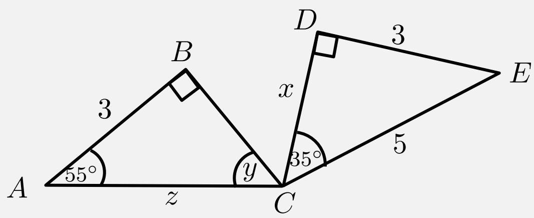


If $\triangle ABC$ is right-angled with $\hat{B} = 90°$, then $b^2 = a^2 + c^2$.
Converse: If $b^2 = a^2 + c^2$, then $\triangle ABC$ is right-angled with $\hat{B} = 90°$.

Example 2: *Triangles*

QUESTION

Determine if the two triangles are congruent. Use the result to find x, $\hat{y}$ and z.



SOLUTION

Step 1 : ***Examine the information given for both triangles***

Step 2 : ***Determine whether $\triangle CDE \equiv \triangle CBA$***

In $\triangle CDE$,

$$\hat{D} + \hat{C} + \hat{E} = 180° \quad \text{(sum of } \angle\text{'s of } \triangle\text{)}$$
$$90° + 35° + \hat{E} = 180°$$
$$\therefore \hat{E} = 55°$$

In $\triangle CDE$ and $\triangle CBA$,

$$
\begin{aligned}
D\hat{E}C = B\hat{A}C = 55^\circ & \quad \text{(proved)} \\
C\hat{D}E = C\hat{B}A = 90^\circ & \quad \text{(given)} \\
DE = BA = 3 & \quad \text{(given)} \\
\therefore \triangle CDE \equiv \triangle CBA & \quad \text{(AAS)}
\end{aligned}
$$

Step 3 : ***Determine the unknown angles and sides***

In $\triangle CDE$,

$$
\begin{aligned}
CE^2 &= DE^2 + CD^2 \quad \text{(Pythagoras)} \\
5^2 &= 3^2 + x^2 \\
x^2 &= 16 \\
\therefore x &= 4
\end{aligned}
$$

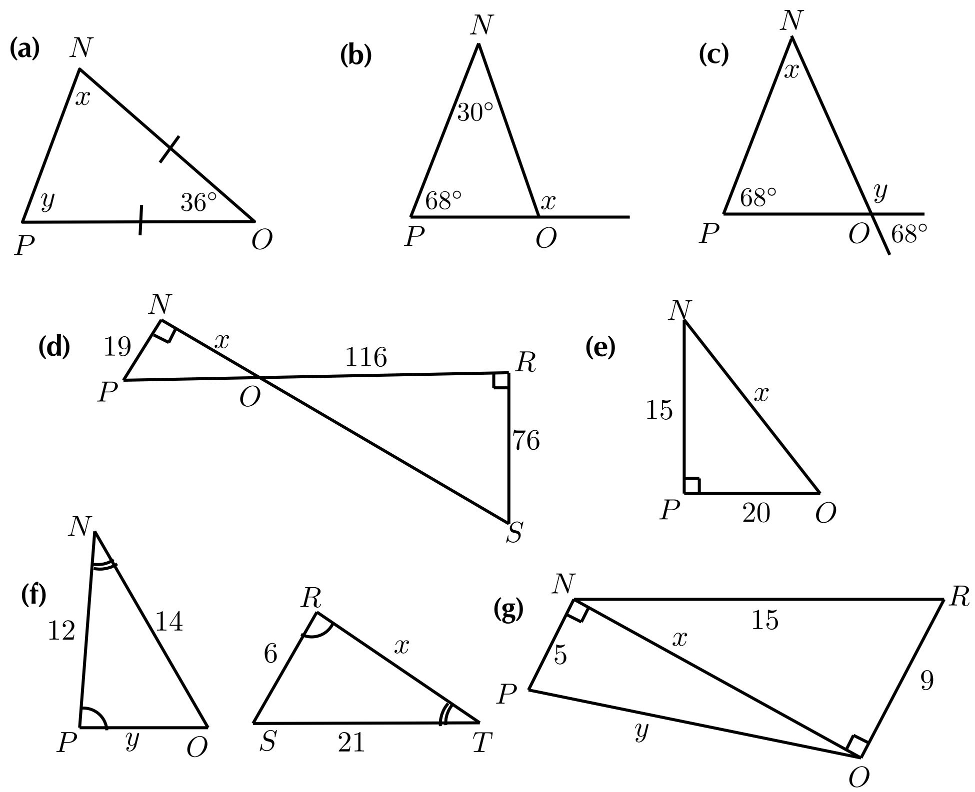
In $\triangle CBA$,

$$
\begin{aligned}
\hat{B} + \hat{A} + \hat{y} &= 180^\circ \quad \text{(sum of } \angle\text{'s of } \triangle) \\
90^\circ + 55^\circ + \hat{y} &= 180^\circ \\
\therefore \hat{y} &= 35^\circ
\end{aligned}
$$

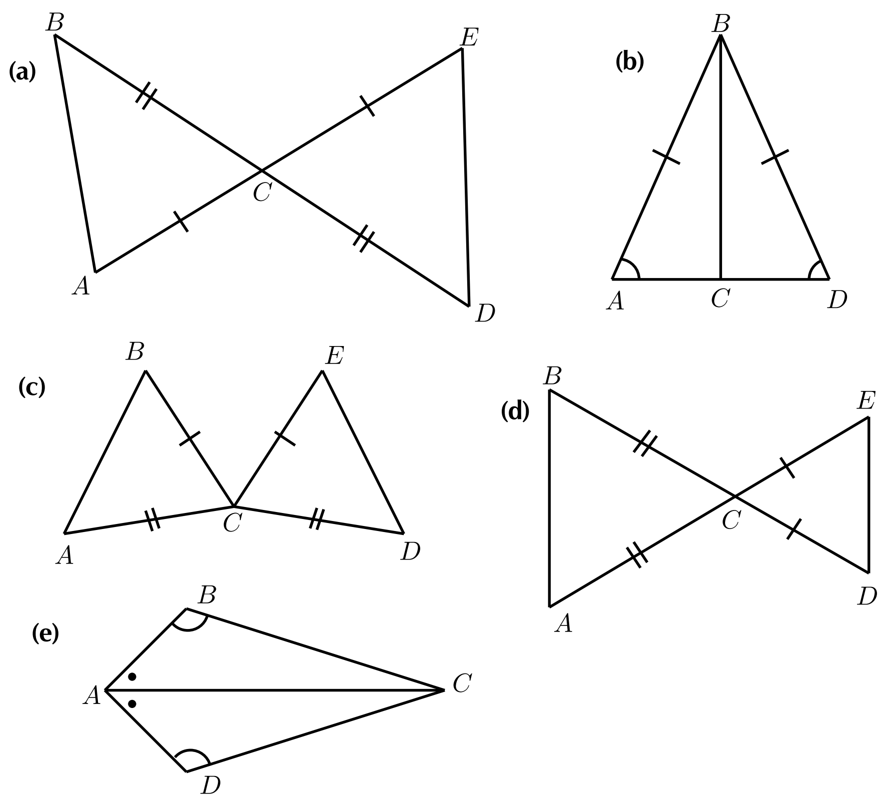
$$
\begin{aligned}
\triangle CDE \equiv \triangle CBA & \quad \text{(proved)} \\
\therefore CE = CA & \\
\therefore z = 5 &
\end{aligned}
$$

Exercise 11 - 2

1. Calculate the unknown variables in each of the following figures.



2. State whether the following pairs of triangles are congruent or not. Give reasons for your answers. If there is not enough information to make a decision, explain why.



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11.1 Quadrilaterals EMAEE

DEFINITION: Quadrilateral

A quadrilateral is a closed shape consisting of four straight line segments.

Video: VMcjo at www.everythingmaths.co.za

Parallelogram EMAEF

DEFINITION: Parallelogram

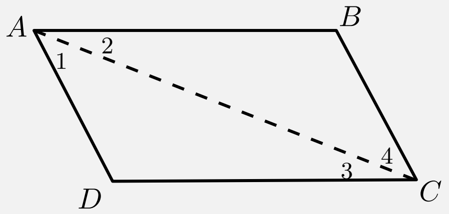
A parallelogram is a quadrilateral with both pairs of opposite sides parallel.

Example 3: *Properties of a parallelogram*

QUESTION

$ABCD$ is a parallelogram with $AB \parallel DC$ and $AD \parallel BC$. Show that:

1. $AB = DC$ and $AD = BC$
2. $\hat{A} = \hat{C}$ and $\hat{B} = \hat{D}$



SOLUTION

Step 1 : **Connect AC to form $\triangle ABC$ and $\triangle CDA$**

Step 2 : **Use properties of parallel lines to indicate all equal angles on the diagram**

Step 3 : **Prove $\triangle ABC \equiv \triangle CDA$**

In $\triangle ABC$ and $\triangle CDA$,

$\hat{A}_2 = \hat{C}_3$ (alt. int. $\angle$'s, $AB \parallel DC$)

$\hat{C}_4 = \hat{A}_1$ (alt. int. $\angle$'s, $BC \parallel AD$)

AC is a common side

$\therefore \triangle ABC \equiv \triangle CDA$ (AAS)

$\therefore \; AB = CD$ and $BC = DA$

$\therefore$ Opposite sides of a parallelogram have equal length.
We have already shown $\hat{A}_2 = \hat{C}_3$ and $\hat{A}_1 = \hat{A}_4$. Therefore,

$$\hat{A} = \hat{A}_1 + \hat{A}_2 = \hat{C}_3 + \hat{C}_4 = \hat{C}.$$

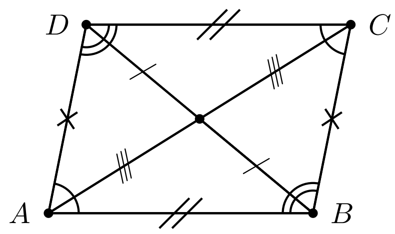
Furthermore,

$$\hat{B} = \hat{D} \quad (\triangle ABC \equiv \triangle CDA)$$

Therefore opposite angles of a parallelogram are equal.

Summary of the properties of a parallelogram:

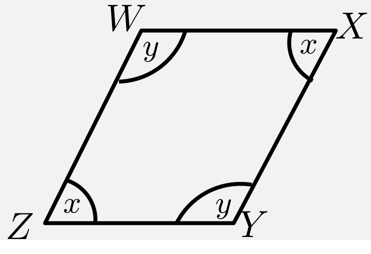
- Both pairs of opposite sides are parallel.
- Both pairs of opposite sides are equal in length.
- Both pairs of opposite angles are equal.
- Both diagonals bisect each other.



Example 4: *Proving a quadrilateral is a parallelogram*

QUESTION

Prove that if both pairs of opposite angles in a quadrilateral are equal, the quadrilateral is a parallelogram.



SOLUTION

Step 1 : ***Find the relationship between $\hat{x}$ and $\hat{y}$***

In $WXYZ$

$$\begin{aligned}
\hat{W} = \hat{Y} &= \hat{y} && \text{(given)}\\
\hat{Z} = \hat{X} &= \hat{x} && \text{(given)}\\
\hat{W} + \hat{X} + \hat{Y} + \hat{Z} &= 360° && \text{(sum of int. } \angle\text{'s of quad.)}\\
\therefore 2\hat{x} + 2\hat{y} &= 360°\\
\therefore \hat{x} + \hat{y} &= 180°\\
\hat{W} + \hat{Z} &= \hat{x} + \hat{y}\\
&= 180°
\end{aligned}$$

But these are co-interior angles between lines WX and ZY. Therefore $WX \parallel ZY$.

Step 2 : ***Find parallel lines***

Similarly $\hat{W} + \hat{X} = 180°$. These are co-interior angles between lines XY and WZ. Therefore $XY \parallel WZ$.

Both pairs of opposite sides of the quadrilateral are parallel, therefore $WXYZ$ is a parallelogram.

Investigation:	*Proving a quadrilateral is a parallelogram*

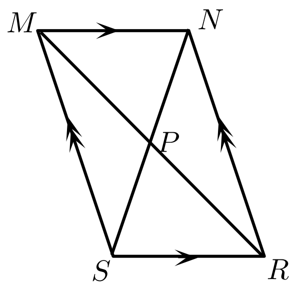
1. Prove that if both pairs of opposite sides of a quadrilateral are equal, then the quadrilateral is a parallelogram.
2. Prove that if the diagonals of a quadrilateral bisect each other, then the quadrilateral is a parallelogram.
3. Prove that if one pair of opposite sides of a quadrilateral are both equal and parallel, then the quadrilateral is a parallelogram.

A quadrilateral is a parallelogram if:

- Both pairs of opposite sides are parallel.
- Both pairs of opposite sides are equal.
- Both pairs of opposite angles are equal.
- The diagonals bisect each other.
- One pair of opposite sides are both equal and parallel.

Exercise 11 - 3

1. Prove that the diagonals of parallelogram $MNRS$ bisect one another at P.



Hint: Use congruency.

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Rectangle

EMAEG

DEFINITION: Rectangle

A rectangle is a parallelogram that has all four angles equal to $90°$.

A rectangle has all the properties of a parallelogram:

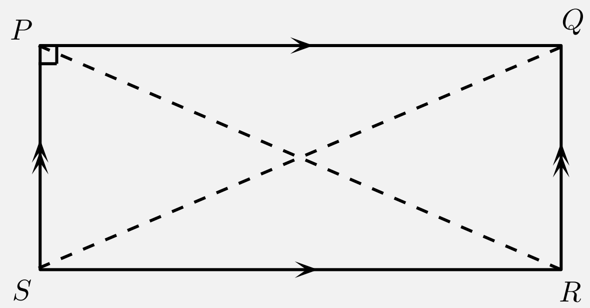
- Both pairs of opposite sides are parallel.
- Both pairs of opposite sides are equal in length.
- Both pairs of opposite angles are equal.
- Both diagonals bisect each other.

It also has the following special property:

Example 5: *Special property of a rectangle*

QUESTION

$PQRS$ is a rectangle. Prove that the diagonals are of equal length.



SOLUTION

Step 1 : ***Connect P to R and Q to S to form $\triangle PSR$ and $\triangle QRS$***

Step 2 : ***Use the definition of a rectangle to fill in on the diagram all equal angles and sides***

Step 3 : ***Prove $\triangle PSR \equiv \triangle QRS$***

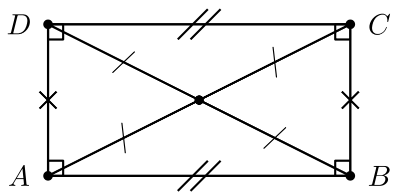
In $\triangle PSR$ and $\triangle QRS$

$PS = QR$	(equal opp. sides of rectangle)
SR is a common side	
$P\hat{S}R = Q\hat{R}S = 90°$	(int. $\angle$ of rectangle)
$\therefore \triangle PSR \equiv \triangle QRS$	(RHS)
Therefore $PR = QS$	

The diagonals of a rectangle are of equal length.

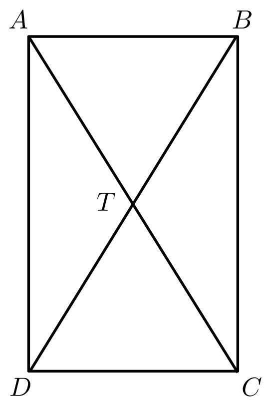
Summary of the properties of a rectangle:

- Both pairs of opposite sides are parallel.
- Both pairs of opposite sides are of equal length.
- Both pairs of opposite angles are equal.
- Both diagonals bisect each other.
- Diagonals are equal in length.
- All interior angles are equal to $90°$.



Exercise 11 - 4

1. $ABCD$ is a quadrilateral. Diagonals AC and BD intersect at T. $AC = BD$, $AT = TC$, $DT = TB$. Prove that:
   (a) $ABCD$ is a parallelogram.
   (b) $ABCD$ is a rectangle.



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(1.) 00gs

Rhombus

EMAEH

DEFINITION: Rhombus

A rhombus is a parallelogram with all four sides of equal length.

A rhombus has all the properties of a parallelogram:

- Both pairs of opposite sides are parallel.
- Both pairs of opposite sides are equal in length.
- Both pairs of opposite angles are equal.
- Both diagonals bisect each other.

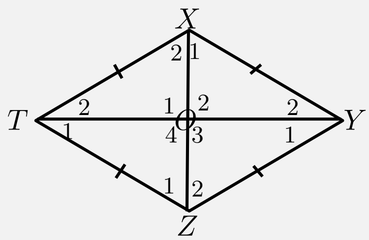
It also has two special properties:

Example 6: *Special properties of a rhombus*

QUESTION

$XYZT$ is a rhombus. Prove that:

1. *the diagonals bisect each other perpendicularly;*
2. *the diagonals bisect the interior angles.*



SOLUTION

Step 1 : ***Use the definition of a rhombus to fill in on the diagram all equal angles and sides***

Step 2 : ***Prove*** $\triangle XTO \equiv \triangle ZTO$

$XT = ZT$	(all sides equal for rhombus)
TO is a common side	
$XO = OZ$	(diag. of rhombus bisect)
$\therefore \triangle XTO \equiv \triangle ZTO$	(SSS)
$\therefore \hat{O}_1 = \hat{O}_4$	
But $\hat{O}_1 + \hat{O}_4 = 180°$	(sum $\angle$'s on str. line)
$\therefore \hat{O}_1 = \hat{O}_4 = 90°$	

We can further conclude that $\hat{O}_1 = \hat{O}_2 = \hat{O}_3 = \hat{O}_4 = 90°$.
Therefore the diagonals bisect each other perpendicularly.

Step 3 : ***Use properties of congruent triangles to prove diagonals bisect interior angles***

$$\hat{X}_2 = \hat{Z}_1 \quad (\triangle XTO \equiv \triangle ZTO)$$

$$\text{and } \hat{X}_2 = \hat{Z}_2 \quad (\text{alt. int. } \angle\text{'s, } XT \parallel YZ)$$

$$\therefore \hat{Z}_1 = \hat{Z}_2$$

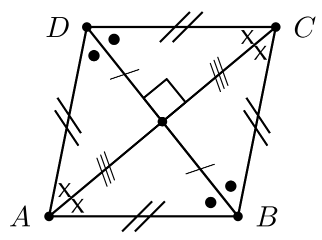
Therefore diagonal XZ bisects $\hat{Z}$. Similarly, we can show that XZ also bisects $\hat{X}$; and that diagonal TY bisects $\hat{T}$ and $\hat{Y}$.
We conclude that the diagonals of a rhombus bisect the interior angles.

To prove a parallelogram is a rhombus, we need to show any one of the following:

- All sides are equal in length.
- Diagonals intersect at right angles.
- Diagonals bisect interior angles.

Summary of the properties of a rhombus:

- Both pairs of opposite sides are parallel.
- Both pairs of opposite sides are equal in length.
- Both pairs of opposite angles are equal.
- Both diagonals bisect each other.
- All sides are equal in length.
- The diagonals bisect each other at $90°$.
- The diagonals bisect both pairs of opposite angles.



Square

EMAEI

DEFINITION: Square

A square is a rhombus with all four interior angles equal to $90°$.

OR

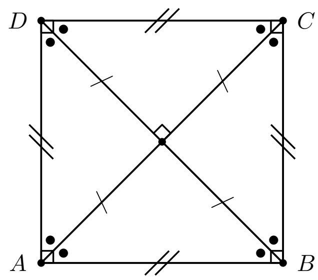
A square is a rectangle with all four sides equal in length.

A square has all the properties of a rhombus:

- Both pairs of opposite sides are parallel.
- Both pairs of opposite sides are equal in length.
- Both pairs of opposite angles are equal.
- Both diagonals bisect each other.
- All sides are equal in length.
- The diagonals bisect each other at $90°$.
- The diagonals bisect both pairs of opposite angles.

It also has the following special properties:

- All interior angles equal $90°$.
- Diagonals are equal in length.
- Diagonals bisect both pairs of interior opposite angles (i.e. all are $45°$).



To prove a parallelogram is a square, we need to show either one of the following:

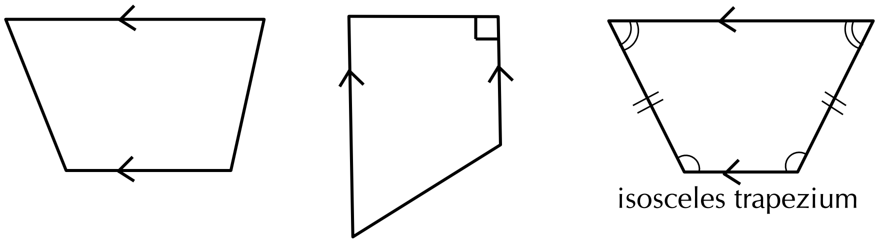
- It is a rhombus (all four sides of equal length) with interior angles equal to $90°$.
- It is a rectangle (interior angles equal to $90°$) with all four sides of equal length.

Trapezium

EMAEJ

DEFINITION: Trapezium

A trapezium is a quadrilateral with one pair of opposite sides parallel.



Kite

EMAEK

DEFINITION: Kite

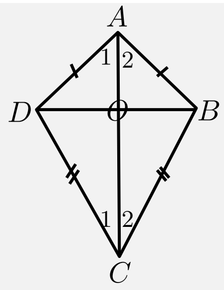
A kite is a quadrilateral with two pairs of adjacent sides equal.

Example 7: *Properties of a kite*

QUESTION

$ABCD$ is a kite with $AD = AB$ and $CD = CB$. Prove that:

1. $A\hat{D}C = A\hat{B}C$
2. *Diagonal AC bisects $\hat{A}$ and $\hat{C}$*



SOLUTION

Step 1 : **Prove** $\triangle ADC \equiv \triangle ABC$

In $\triangle ADC$ and $\triangle ABC$,

$$AD = AB \quad \text{(given)}$$

$$CD = CB \quad \text{(given)}$$

$$AC \text{ is a common side}$$

$$\therefore \triangle ADC \equiv \triangle ABC \quad \text{(SSS)}$$

$$\therefore A\hat{D}C = A\hat{B}C$$

Therefore one pair of opposite angles are equal in kite $ABCD$.

Step 2 : **Use properties of congruent triangles to prove AC bisects $\hat{A}$ and $\hat{C}$**

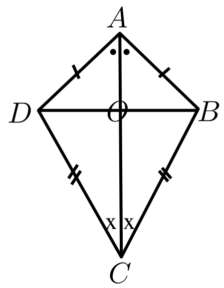
$$\hat{A}_1 = \hat{A}_2 \quad (\triangle ADC \equiv \triangle ABC)$$

$$\text{and } \hat{C}_1 = \hat{C}_2 \quad (\triangle ADC \equiv \triangle ABC)$$

Therefore diagonal AC bisects $\hat{A}$ and $\hat{C}$.

We conclude that the diagonal between the equal sides of a kite bisects the two interior angles and is an axis of symmetry.

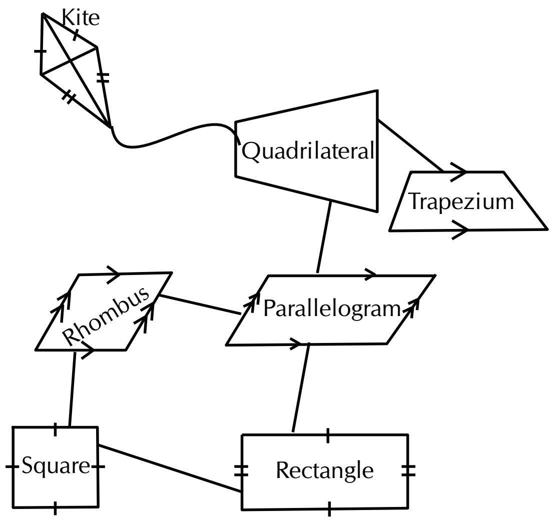
Summary of the properties of a kite:



- Diagonal between equal sides bisects the other diagonal.
- One pair of opposite angles are equal (the angles between unequal sides).
- Diagonal between equal sides bisects the interior angles and is an axis of symmetry.
- Diagonals intersect at $90°$.

Investigation:	*Relationships between the different quadrilaterals*

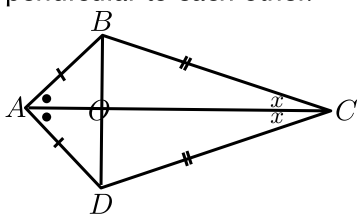
Heather has drawn the following diagram to illustrate her understanding of the relationships between the different quadrilaterals. The following diagram summarises the different types of special quadrilaterals.



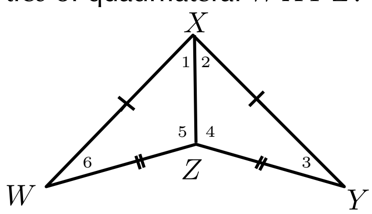
1. Explain her possible reasoning for structuring the diagram as shown.
2. Design your own diagram to show the relationships between the different quadrilaterals and write a short explanation of your design.

Exercise 11 - 5

1. Use the sketch of quadrilateral $ABCD$ to prove the diagonals are perpendicular to each other.



2. Explain why quadrilateral $WXYZ$ is a kite. Write down all the properties of quadrilateral $WXYZ$.



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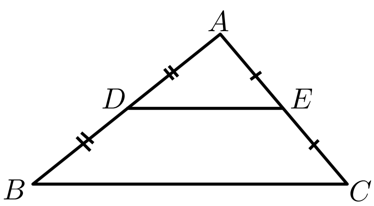
11.2 The mid-point theorem

EMAEL

Investigation:	*Proving the mid-point theorem*



1. Draw a large scalene triangle on a sheet of paper.
2. Name the vertices A, B and C. Find the mid-points (D and E) of two sides

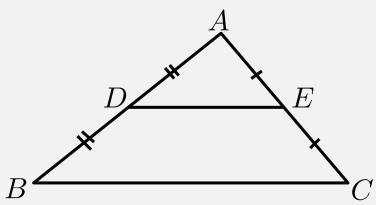
and connect them.

3. Cut out $\triangle ABC$ and cut along line DE.
4. Place $\triangle ADE$ on quadrilateral $BDEC$ with vertex E on vertex C. Write down your observations.
5. Shift $\triangle ADE$ to place vertex D on vertex B. Write down your observations.
6. What do you notice about the lengths DE and BC?
7. Make a conjecture regarding the line joining the mid-point of two sides of a triangle.

Example 8: *Mid-point theorem*

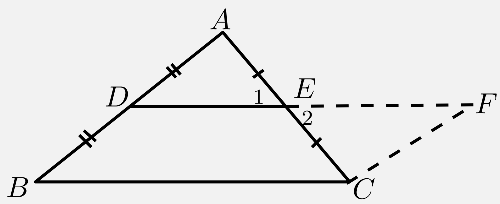
QUESTION

Prove that the line joining the mid-points of two sides of a triangle is parallel to the third side and equal to half the length of the third side.



SOLUTION

Step 1 : **Extend DE to F so that $DE = EF$ and join FC**



Step 2 : **Prove $BCFD$ is a parallelogram**

In $\triangle EAD$ and $\triangle ECF$:

$$\hat{E}_1 = \hat{E}_2 \quad \text{(vert. opp. } \angle\text{'s)}$$

$$AE = CE \quad \text{(given)}$$

$$DE = EF \quad \text{(by construction)}$$

$$\therefore \triangle EAD \equiv \triangle ECF \quad \text{(SAS)}$$

$$\therefore A\hat{D}E = C\hat{F}E$$

But these are alternate interior angles, therefore $BD \parallel FC$

$$BD = DA \quad \text{(given)}$$

$$DA = FC \quad (\triangle EAD \equiv \triangle ECF)$$

$$\therefore BD = FC$$

$\therefore BCFD$ is a parallelogram (one pair opp. sides $=$ and $\parallel$)

Therefore $DE \parallel BC$.
We conclude that the line joining the two mid-points of two sides of a triangle is parallel to the third side.

Step 3 : ***Use properties of parallelogram** $BCFD$ **to prove that** $DE = \frac{1}{2}BC$*

$$DF = BC \quad \text{(opp. sides parm equal)}$$

and $DF = 2(DE)$ (by construction)

$$\therefore 2DE = BC$$

$$\therefore DE = \tfrac{1}{2}BC$$

We conclude that the line joining the mid-point of two sides of a triangle is equal to half the length of the third side.

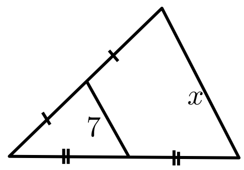
Converse

The converse of this theorem states: If a line is drawn through the mid-point of a side of a triangle parallel to the second side, it will bisect the third side.

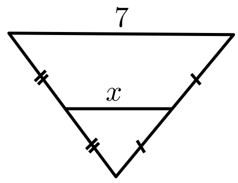
Exercise 11 - 6

1. Find x and y in the following:

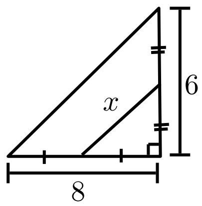
(a)



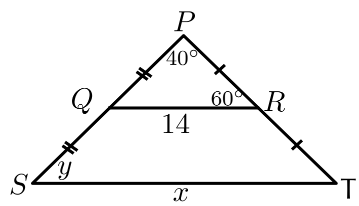
(b)



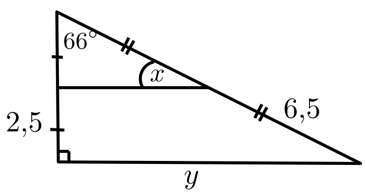
(c)



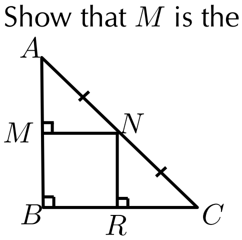
(d)



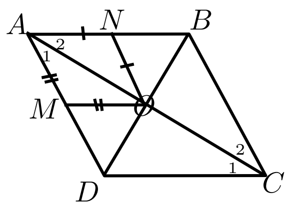
(e)



2. Show that M is the mid-point of AB and that $MN = RC$.



3. $ABCD$ is a rhombus with $AM = MO$ and $AN = NO$. Prove $ANOM$ is also a rhombus.



More practice video solutions or help at www.everythingmaths.co.za

(1.) 00gv (2.) 00gw (3.) 00mj

11.3 Proofs and conjectures

EMAEM

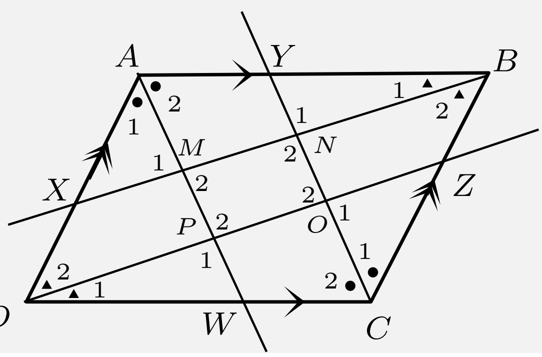
You can use geometry and the properties of polygons to find unknown lengths and angles in various quadrilaterals and triangles. The following worked example will help make this clearer.

Video: VMckg at www.everythingmaths.co.za

Example 9: *Proving a quadrilateral is a parallelogram*

QUESTION

In parallelogram $ABCD$, the bisectors of the angles (AW, BX, CY and DZ) have been constructed. You are also given $AB = CD$, $AD = BC$, $AB \parallel CD$, $AD \parallel BC$, $\hat{A} = \hat{C}$, and $\hat{B} = \hat{D}$. Prove that $MNOP$ is a parallelogram.



SOLUTION

Step 1 : ***Use properties of the parallelogram $ABCD$ to fill in on the diagram all equal sides and angles***

Step 2 : ***Prove that $\hat{M}_2 = \hat{O}_2$***

In $\triangle CDZ$ and $\triangle ABX$,

$$D\hat{C}Z = B\hat{A}X \quad \text{(given)}$$

$$\hat{D}_1 = \hat{B}_1 \quad \text{(given)}$$

$$DC = AB \quad \text{(given)}$$

$$\therefore \triangle CDZ \equiv \triangle ABX \quad \text{(AAS)}$$

$$\therefore CZ = AX$$

$$\text{and } C\hat{Z}D = A\hat{X}B$$

In $\triangle XAM$ and $\triangle ZCO$,

$$X\hat{A}M = Z\hat{C}O \quad \text{(given: } \triangle CDZ \equiv \triangle ABX\text{)}$$

$$A\hat{X}M = C\hat{Z}O \quad \text{(proved above)}$$

$$AX = CZ \quad \text{(proved above)}$$

$$\therefore \triangle XAM \equiv \triangle ZCO \quad \text{(AAS)}$$

$$\therefore \hat{M}_1 = \hat{O}_1$$

$$\text{but } \hat{M}_1 = \hat{M}_2 \quad \text{(vert. opp. } \angle\text{'s)}$$

$$\text{and } \hat{O}_1 = \hat{O}_2 \quad \text{(vert. opp. } \angle\text{'s)}$$

$$\therefore \hat{M}_2 = \hat{O}_2$$

Step 3 : ***Similarly, we can show that $\hat{N}_2 = \hat{P}_2$***

First show $\triangle ADW \equiv \triangle CBY$. Then show $\triangle PDW \equiv \triangle NBY$.

Step 4 : ***Conclusion***

Both pairs of opposite angles of $MNOP$ are equal. Therefore $MNOP$ is a parallelogram.

Chapter 11 | Summary

Summary presentation: VMdml at www.everythingmaths.co.za

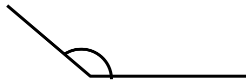
- A quadrilateral is a closed shape consisting of four straight line segments.
- A parallelogram is a quadrilateral with both pairs of opposite sides parallel.
  - Both pairs of opposite sides are equal in length.
  - Both pairs of opposite angles are equal.
  - Both diagonals bisect each other.
- A rectangle is a parallelogram that has all four angles equal to $90°$.
  - Both pairs of opposite sides are parallel.
  - Both pairs of opposite sides are equal in length.
  - The diagonals bisect each other.
  - The diagonals are equal in length.
  - All interior angles are equal to $90°$.
- A rhombus is a parallelogram that has all four sides equal in length.
  - Both pairs of opposite sides are parallel.
  - All sides are equal in length.
  - Both pairs of opposite angles are equal.
  - The diagonals bisect each other at $90°$.
  - The diagonals of a rhombus bisect both pairs of opposite angles.
- A square is a rhombus that has all four interior angles equal to $90°$.
  - Both pairs of opposite sides are parallel.
  - The diagonals bisect each other at $90°$.
  - All interior angles are equal to $90°$.
  - The diagonals are equal in length.
  - The diagonals bisect both pairs of interior opposite angles (i.e. all are $45°$).
- A trapezium is a quadrilateral with one pair of opposite sides parallel.
- A kite is a quadrilateral with two pairs of adjacent sides equal.
  - One pair of opposite angles are equal (the angles are between unequal sides).
  - The diagonal between equal sides bisects the other diagonal.
  - The diagonal between equal sides bisects the interior angles.
  - The diagonals intersect at $90°$.

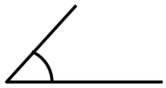
- The mid-point theorem: The line joining the mid-points of two sides of a triangle is parallel to the third side and equal to half the length of the third side.

Chapter 11 End of Chapter Exercises

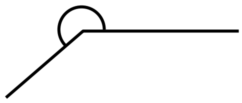
1. Identify the types of angles shown below:

(a) 

(b) 

(c) 

(d) 

(e) 

(f) An angle of $91°$

(g) An angle of $180°$

(h) An angle of $210°$

2. Assess whether the following statements are true or false. If the statement is false, explain why:

   (a) A trapezium is a quadrilateral with two pairs of opposite sides that are parallel.

   (b) Both diagonals of a parallelogram bisect each other.

   (c) A rectangle is a parallelogram that has one corner angles equal to $90°$.

   (d) Two adjacent sides of a rhombus have different lengths.

   (e) The diagonals of a kite intersect at right angles.

   (f) All squares are parallelograms.

   (g) A rhombus is a kite with a pair of equal, opposite sides.

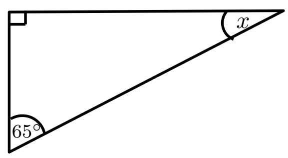
   (h) The diagonals of a parallelogram are axes of symmetry.

   (i) The diagonals of a rhombus are equal in length.

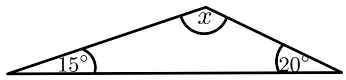
   (j) Both diagonals of a kite bisect the interior angles.

3. Calculate x and y in the diagrams below:

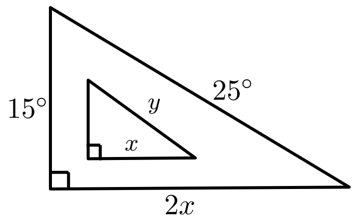
(a)



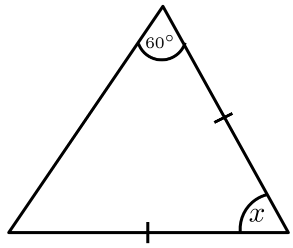
(b)



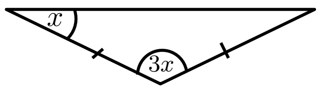
(c)



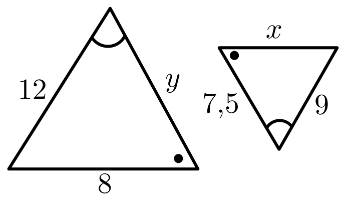
(d)



(e)

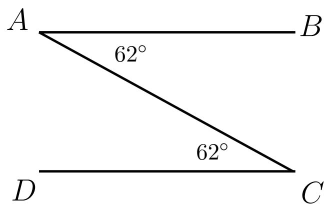


(f)

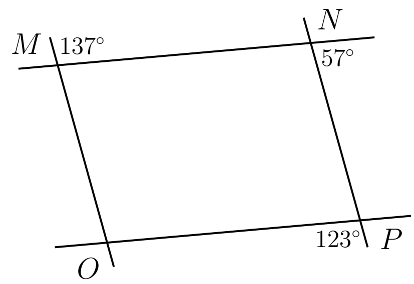


4. Find all the pairs of parallel lines in the following figures, giving reasons in each case.

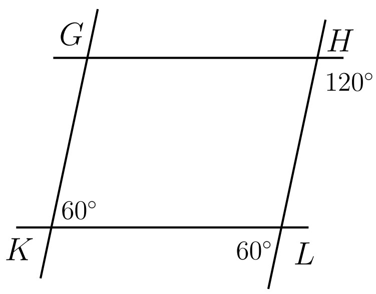
(a)



(b)

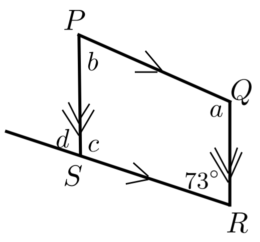


(c)

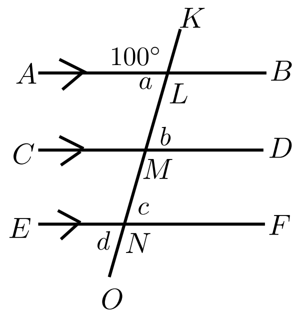


5. Find angles a, b, c and d in each case, giving reasons:

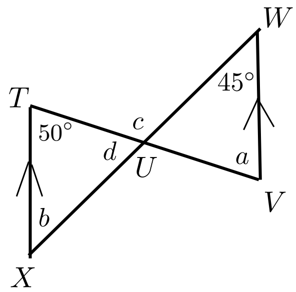
(a)



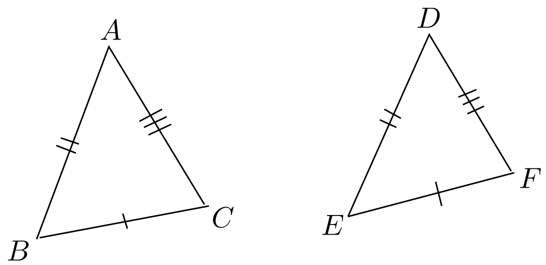
(b)



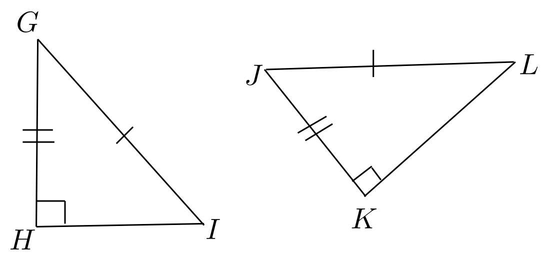
(c)



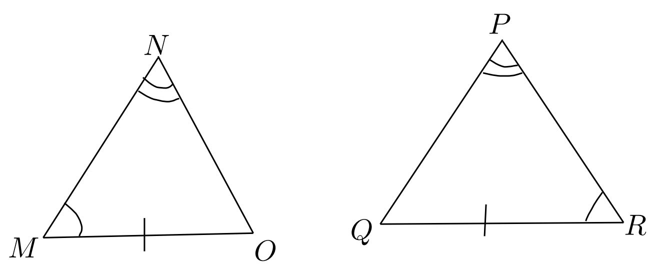
6. Say which of the following pairs of triangles are congruent with reasons.



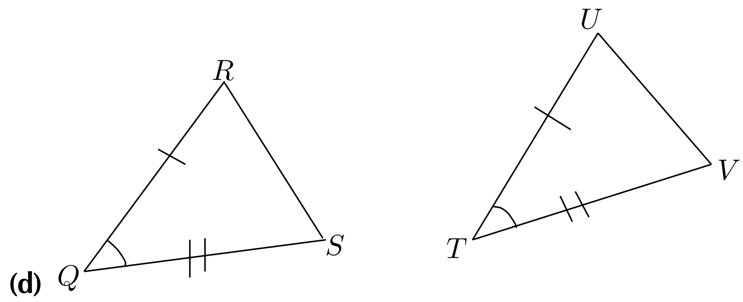
(a)



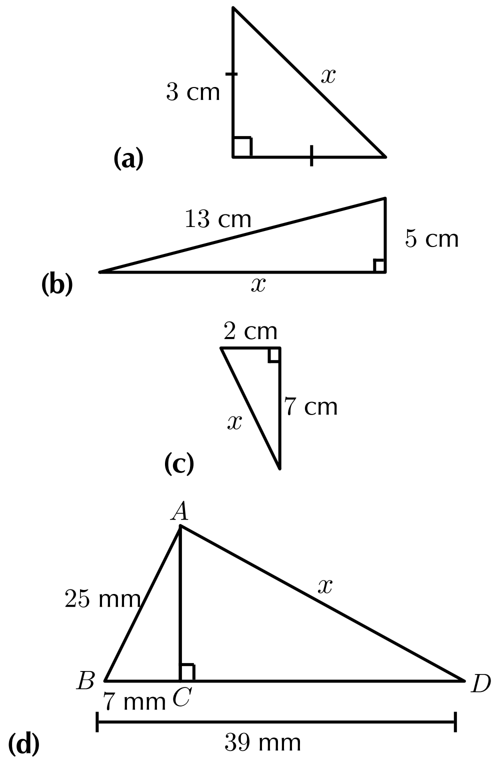
(b)



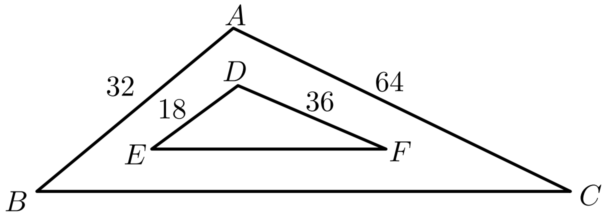
(c)



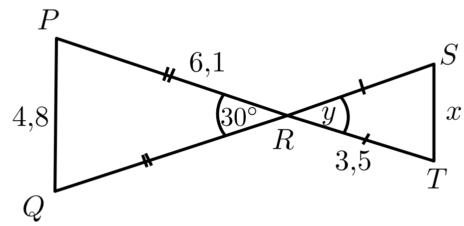
7. Using Pythagoras' theorem for right-angled triangles, calculate the length x:



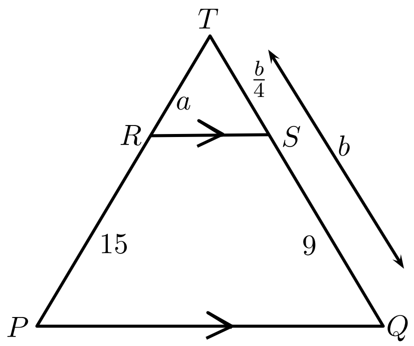
8. Consider the diagram below. Is $\triangle ABC \mathrel{|||} \triangle DEF$? Give reasons for your answer.



9. Explain why $\triangle PQR$ is similar to $\triangle TSR$ and calculate the values of x and y.



10. Calculate a and b:

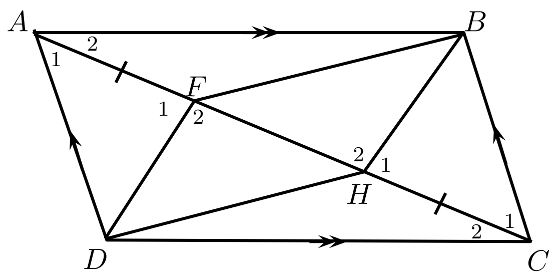


11. $ABCD$ is a parallelogram with diagonal AC.
Given that $AF = HC$, show that:

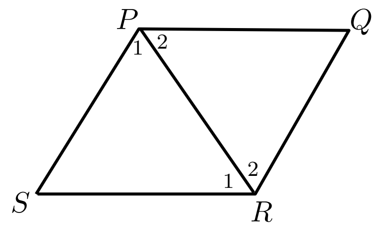
(a) $\triangle AFD \equiv \triangle CHB$

(b) $DF \parallel HB$

(c) $DFBH$ is a parallelogram



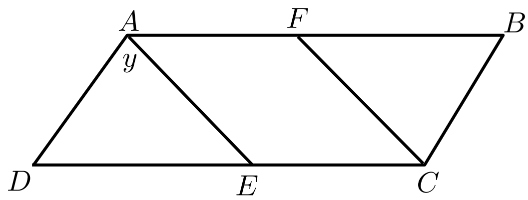
12. $\triangle PQR$ and $\triangle PSR$ are equilateral triangles. Prove that $PQRS$ is a rhombus.



13. Given parallelogram $ABCD$ with AE and FC, AE bisecting $\hat{A}$ and FC bisecting $\hat{C}$:

    (a) Write all interior angles in terms of y.

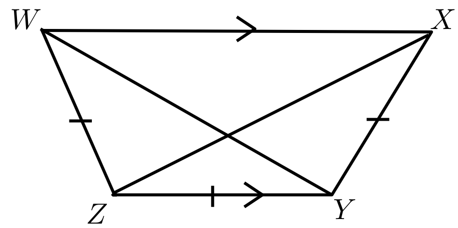
    (b) Prove that $AFCE$ is a parallelogram.



14. Given that $WZ = ZY = YX$, $\hat{W} = \hat{X}$ and $WX \parallel ZY$, prove that:

    (a) XZ bisects $\hat{X}$

    (b) $WY = XZ$

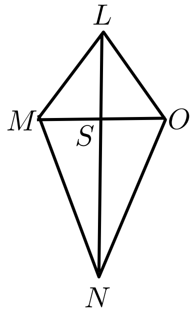


15. $LMNO$ is a quadrilateral with $LM = LO$ and diagonals that intersect at S such that $MS = SO$. Prove that:

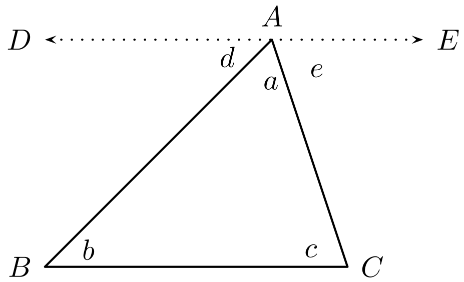
    (a) $M\hat{L}S = S\hat{L}O$

    (b) $\triangle LON \equiv \triangle LMN$

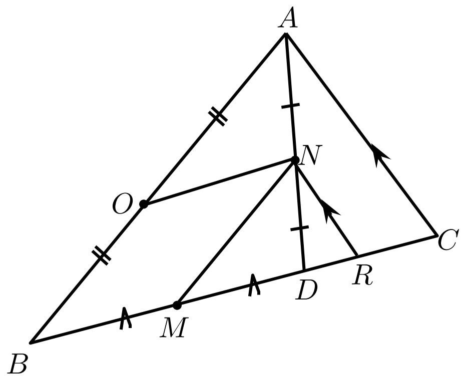
    (c) $MO \perp LN$



16. Using the figure below, show that the sum of the three angles in a triangle is 180°. Line DE is parallel to BC.



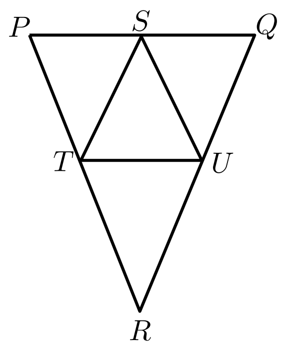
17. D is a point on BC, in $\triangle ABC$. N is the mid-point of AD. O is the mid-point of AB and M is the mid-point of BD. $NR \parallel AC$.



(a) Prove that $OBMN$ is a parallelogram.

(b) Prove that $BC = 2MR$.

18. PQR is an isosceles with $PR = QR$. S is the mid-point of PQ, T is the mid-point of PR and U is the mid-point of RQ.

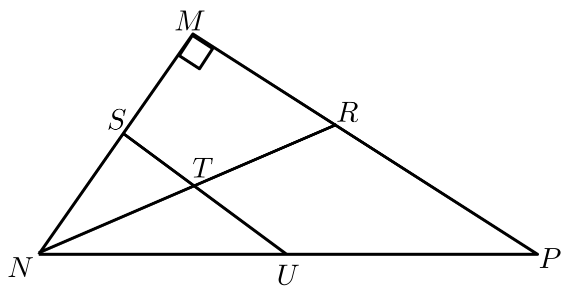


(a) Prove $\triangle STU$ is also isosceles.

(b) What type of quadrilateral is $STRU$? Motivate your answer.

(c) If $R\hat{T}U = 68°$ calculate, with reasons, the size of $T\hat{S}U$.

19. In $\triangle MNP$, $M = 90°$, S is the mid-point of MN and T is the mid-point of NR.



(a) Prove U is the mid-point of NP.

(b) If $ST = 4$ cm and the area of $\triangle SNT$ is 6 cm^2, calculate the area of $\triangle MNR$.

(c) Prove that the area of $\triangle MNR$ will always be four times the area of $\triangle SNT$, let $ST = x$ units and $SN = y$ units.

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(1.) 00gx	(2.) 00gy	(3.) 00gz	(4.) 00h0	(5.) 00h1
(6.) 00h2	(7.) 00h3	(8.) 00h4	(9.) 00h5	(10.) 00h6
(11.) 00h7	(12.) 00h8	(13.) 00h9	(14.) 00ha	(15.) 00hb
(16.) 00hc	(17.) 023f	(18.) 023g	(19.) 023h	

Measurements 12

This chapter examines the surface areas of two dimensional objects and volumes of three dimensional objects, otherwise known as solids. In order to work with these objects, you need to know how to calculate the surface area and perimeter of the two dimensional shapes below.

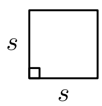
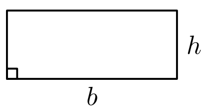
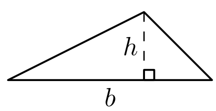
Video: VMcmk at www.everythingmaths.co.za

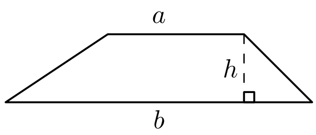
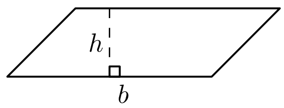
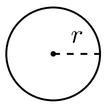
12.1 Area of a polygon

EMAEN

DEFINITION: Area

Area is the two dimensional space inside the boundary of a flat object. It is measured in square units.

Name	**Shape**	**Formula**
Square		Area $= s^2$
Rectangle		Area $= b \times h$
Triangle		Area $= \frac{1}{2} b \times h$

Name	Shape	Formula
Trapezium		Area $= \frac{1}{2}(a+b) \times h$
Parallelogram		Area $= b \times h$
Circle		Area $= \pi r^2$ (Circumference $= 2\pi r$)

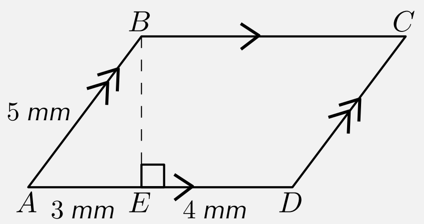
Video: VMdxu at www.everythingmaths.co.za

Video: VMdya at www.everythingmaths.co.za

Example 1: *Finding the area of a polygon*

QUESTION

Find the area of the following parallelogram:



SOLUTION

Step 1 : ***Find the height BE of the parallelogram using the Theorem of Pythagoras***

$$
\begin{aligned}
AB^2 &= BE^2 + AE^2 \\
\therefore BE^2 &= AB^2 - AE^2 \\
&= 5^2 - 3^2 \\
&= 16 \\
\therefore BE &= 4 \text{ mm}
\end{aligned}
$$

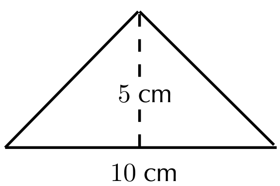
Step 2 : ***Find the area using the formula for a parallelogram***

$$
\begin{aligned}
\text{Area} &= b \times h \\
&= AD \times BE \\
&= 7 \times 4 \\
&= 28 \text{ mm}^2
\end{aligned}
$$

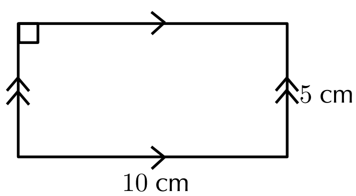
Exercise 12 - 1

Find the areas of each of the polygons below:

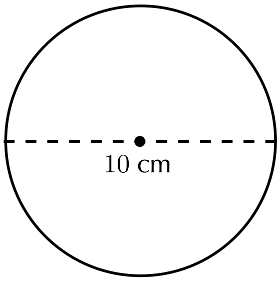
1.



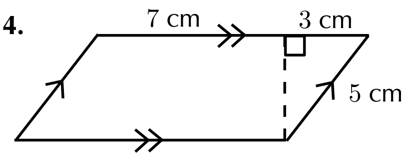
2.



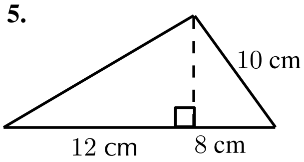
3.



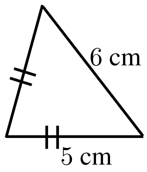
4.



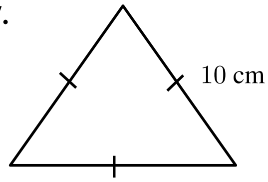
5.



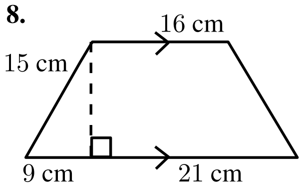
6.



7.



8.



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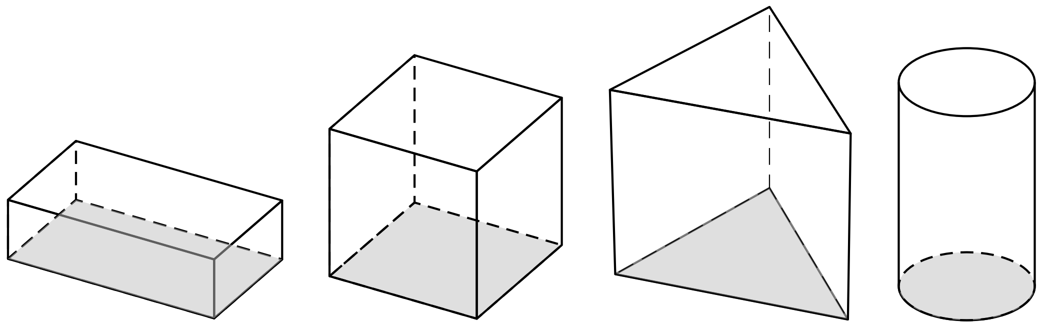
(1 - 8.) 00hi

12.2 Right prisms and cylinders EMAEO

> *DEFINITION: Right prism*
>
> A right prism is a geometric solid that has a polygon as its base and vertical sides perpendicular to the base. The base and top surface are the same shape and size. It is called a "right" prism because the angles between the base and sides are right angles.

A triangular prism has a triangle as its base, a rectangular prism has a rectangle as its base, and a cube is a rectangular prism with all its sides of equal length. A cylinder is another type of right prism which has a circle as its base.

Examples of right prisms are given below: a rectangular prism, a cube, a triangular prism and a cylinder.



Video: VMcph at www.everythingmaths.co.za

Surface area of prisms and cylinders EMAEP

> *DEFINITION: Surface area*
>
> Surface area is the total area of the exposed or outer surfaces of a prism.

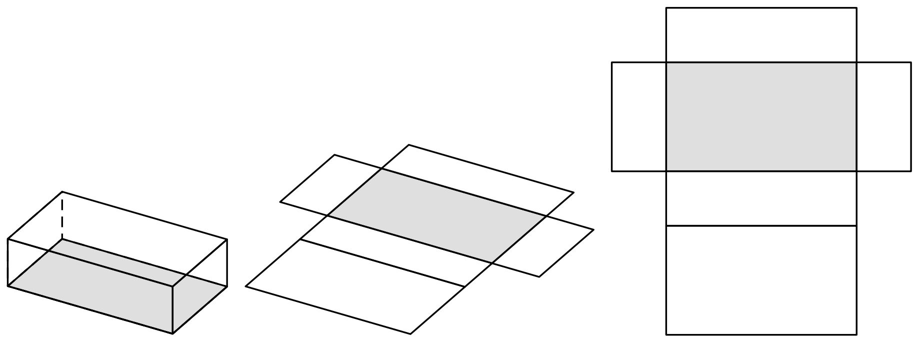
This is easier to understand if we imagine the prism to be a cardboard box that we can unfold. A solid that is unfolded like this is called a net. When a prism is unfolded into a net, we can clearly see each of its faces. In order to calculate the surface area of the prism, we can then simply calculate the area of each face, and add them all together.

For example, when a triangular prism is unfolded into a net, we can see that it has two faces that are triangles and three faces that are rectangles. To calculate the surface area of the prism, we find the area of each triangle and each rectangle, and add them together. In the case of a cylinder the top and bottom faces are circles and the curved surface flattens into a rectangle with a length that is equal to the circumference of the circular base. To calculate the surface area we therefore find the area of the two circles and the rectangle and add them together.

Video: VMcre at www.everythingmaths.co.za

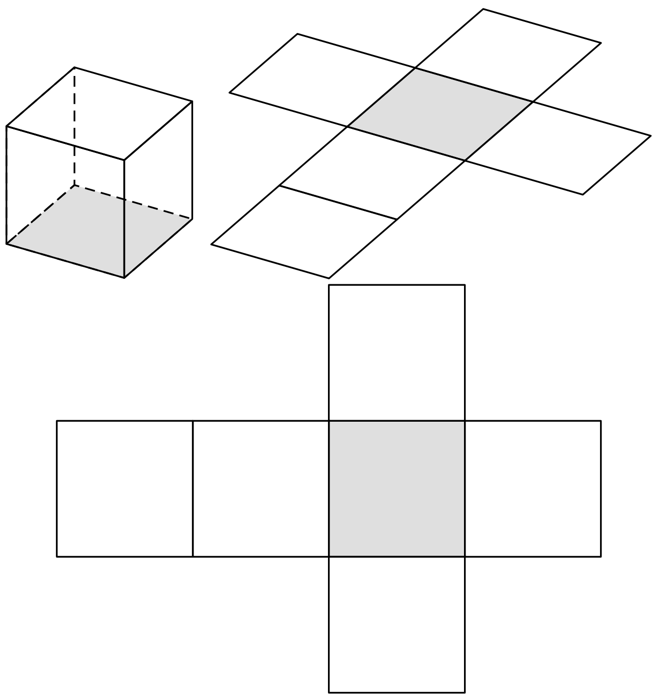
Below are examples of right prisms that have been unfolded into nets:

Rectangular prism



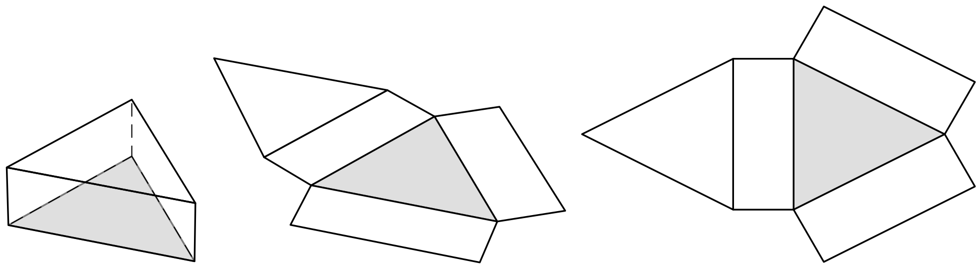
A rectangular prism unfolded into a net is made up of six rectangles.

Cube



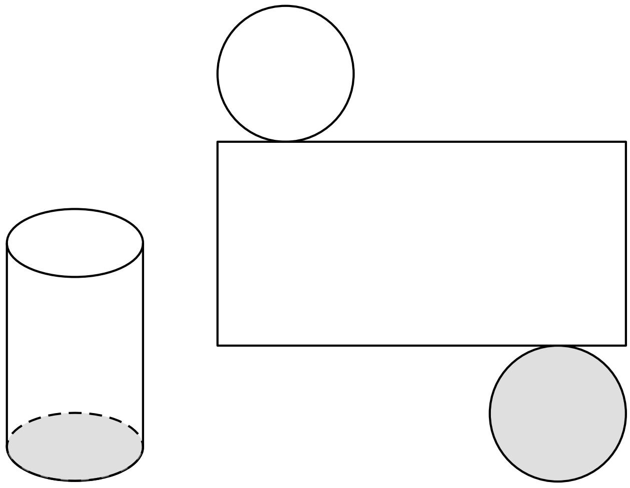
A cube unfolded into a net is made up of six identical squares.

Triangular prism



A triangular prism unfolded into a net is made up of two triangles and three rectangles. The sum of the lengths of the rectangles is equal to the perimeter of the triangles.

Cylinder

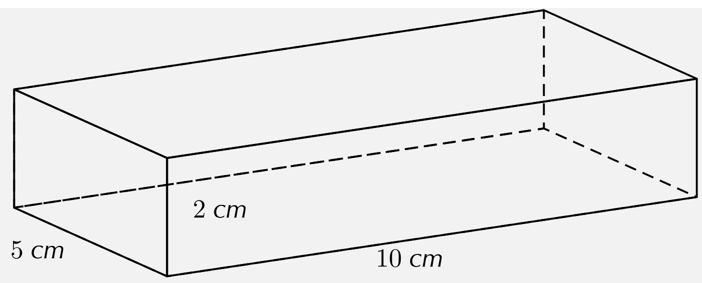


A cylinder unfolded into a net is made up of two identical circles and a rectangle with a length equal to the circumference of the circles.

Example 2: *Finding the surface area of a rectangular prism*

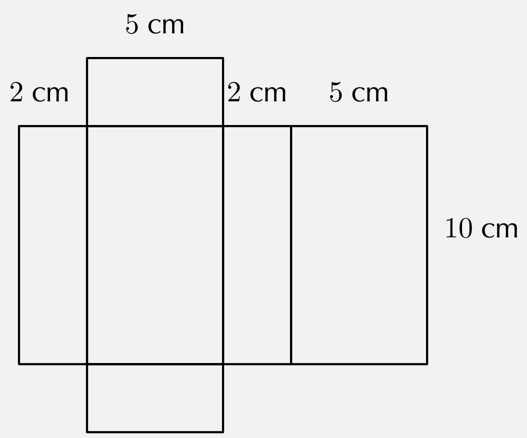
QUESTION

Find the surface area of the following rectangular prism:



SOLUTION

Step 1 : ***Sketch and label the net of the prism***



Step 2 : ***Find the areas of the different shapes in the net***

$$\begin{aligned}\text{large rectangle} &= \text{perimeter of small rectangle} \times \text{length} \\ &= (2+5+2+5) \times 10 \\ &= 14 \times 10 \\ &= 140 \text{ cm}^2\end{aligned}$$

$$\begin{aligned}2 \times \text{small rectangle} &= 2(5 \times 2) \\ &= 2(10) \\ &= 20 \text{ cm}^2\end{aligned}$$

Step 3 : ***Find the sum of the areas of the faces***

large rectangle $+ 2 \times$ small rectangle $= 140 + 20 = 160$ cm^2.

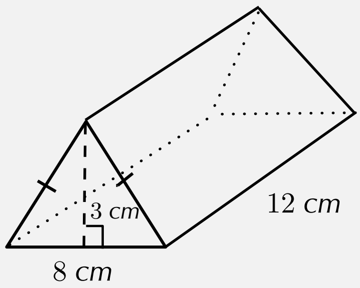
Step 4 : ***Write the final answer***

The surface area of the rectangular prism is 160 cm^2.

Example 3: *Finding the surface area of a triangular prism*

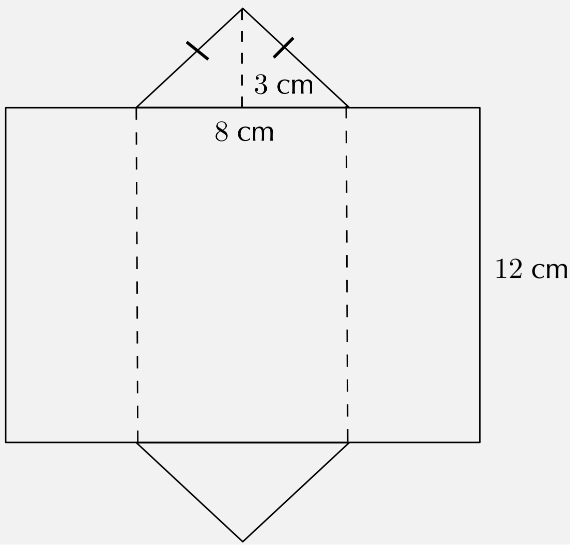
QUESTION

Find the surface area of the following triangular prism:



SOLUTION

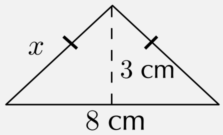
Step 1 : ***Sketch and label the net of the prism***



Step 2 : ***Find the area of the different shapes in the net***

To find the area of the rectangle, we need to calculate its length, which is equal to the perimeter of the triangles.

To find the perimeter of the triangle, we have to first find the length of its sides using the theorem of Pythagoras:



$$\begin{aligned} x^2 &= 3^2 + \left(\frac{8}{2}\right)^2 \\ x^2 &= 3^2 + 4^2 \\ &= 25 \\ \therefore x &= 5 \text{ cm} \\ \therefore \text{perimeter of triangle} &= 5 + 5 + 8 \\ &= 18 \text{ cm} \end{aligned}$$

$$\begin{aligned} \therefore \text{area of large rectangle} &= \text{perimeter of triangle} \times \text{length} \\ &= 18 \times 12 \\ &= 216 \text{ cm}^2 \end{aligned}$$

$$\begin{aligned} \text{area of triangle} &= \frac{1}{2} b \times h \\ &= \frac{1}{2} \times 8 \times 3 \\ &= 12 \text{ cm}^2 \end{aligned}$$

Step 3 : ***Find the sum of the areas of the faces***

$$\begin{aligned} \text{surface area} &= \text{area large rectangle} + (2 \times \text{area of triangle}) \\ &= 216 + 2(12) \\ &= 240 \text{ cm}^2 \end{aligned}$$

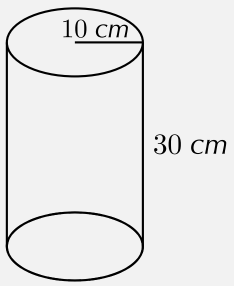
Step 4 : ***Write the final answer***

The surface area of the triangular prism is $240\ \text{cm}^2$.

Example 4: *Finding the surface area of a cylindrical prism*

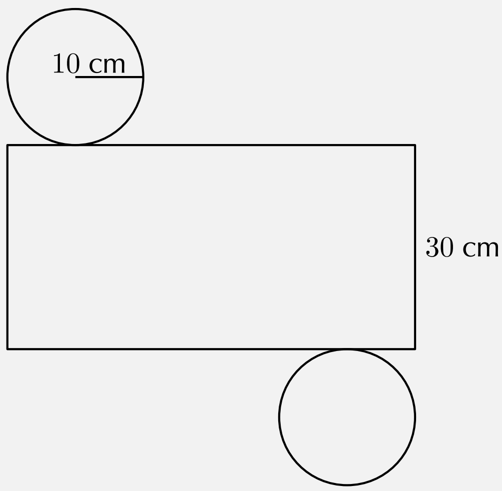
QUESTION

Find the surface area of the following cylinder (correct to 1 decimal place):



SOLUTION

Step 1 : ***Sketch and label the net of the prism***



Step 2 : ***Find the area of the different shapes in the net***

$$\begin{aligned}
\text{area of large rectangle} &= \text{circumference of circle} \times \text{length} \\
&= 2\pi r \times l \\
&= 2\pi(10) \times 30 \\
&= 1\ 884{,}96\ \text{cm}^2
\end{aligned}$$

$$\begin{aligned}
\text{area of circle} &= \pi r^2 \\
&= \pi 10^2 \\
&= 314{,}16\ \text{cm}^2
\end{aligned}$$

$$\begin{aligned}
\text{surface area} &= \text{area large rectangle} + (2 \times \text{area of circle}) \\
&= 1\ 884{,}96 + 2(314{,}16) \\
&= 2\ 513{,}28\ \text{cm}^2
\end{aligned}$$

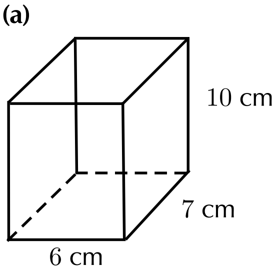
Step 3 : ***Write the final answer***

The surface area of the cylinder is $2\ 513{,}28$ cm^2.

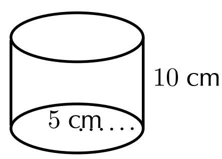
Exercise 12 - 2

1. Calculate the surface area of the following prisms:

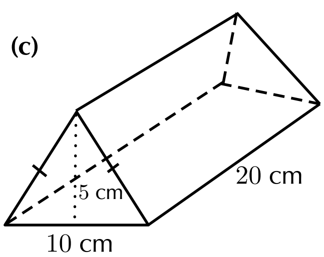
(a)



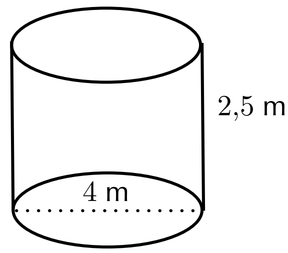
(b)



(c)



2. If a litre of paint covers an area of $2\ \text{m}^2$, how much paint does a painter need to cover:
   (a) a rectangular swimming pool with dimensions 4 m $\times$ 3 m $\times$ $2{,}5$ m (the inside walls and floor only);
   (b) the inside walls and floor of a circular reservoir with diameter 4 m and height $2{,}5$ m.



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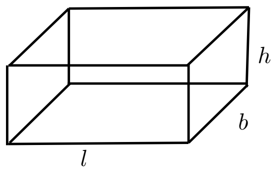
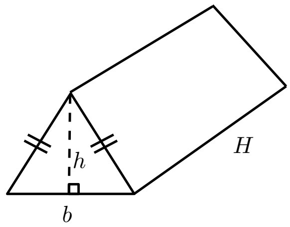
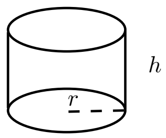
(1.) 00hj (2.) 00hk

Volume of prisms and cylinders EMAEQ

> *DEFINITION: Volume*
>
> Volume is the three dimensional space occupied by an object, or the contents of an object. It is measured in cubic units.

The volume of a right prism is simply calculated by multiplying the area of the base of a solid by the height of the solid.

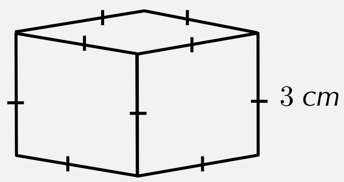
Video: VMcth at www.everythingmaths.co.za

Rectangular prism		Volume $=$ area of base $\times$ height $=$ area of rectangle $\times$ height $= l \times b \times h$
Triangular prism		Volume $=$ area of base $\times$ height $=$ area of triangle $\times$ height $= \left(\frac{1}{2}b \times h\right) \times H$
Cylinder		Volume $=$ area of base $\times$ height $=$ area of circle $\times$ height $= \pi r^2 \times h$

Example 5: *Finding the volume of a cube*

QUESTION

Find the volume of the following cube:



SOLUTION

Step 1 : ***Find the area of the base***

$$\begin{aligned}\text{area of square} &= s^2\\ &= 3^2\\ &= 9\text{ cm}^2\end{aligned}$$

Step 2 : ***Multiply the area of the base by the height of the solid to find the volume***

$$\begin{aligned}\text{volume} &= \text{area of base} \times \text{height}\\ &= 9 \times 3\\ &= 27\text{ cm}^3\end{aligned}$$

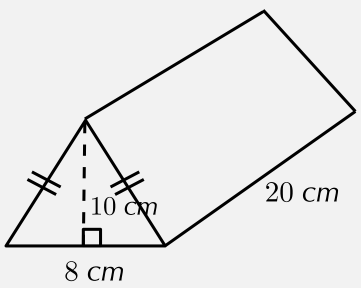
Step 3 : ***Write the final answer***

The volume of the cube is 27 cm^3.

Example 6: *Finding the volume of a triangular prism*

QUESTION

Find the volume of the triangular prism:



SOLUTION

Step 1 : ***Find the area of the base***

$$
\begin{aligned}
\text{area of triangle} &= \frac{1}{2}b \times h \\
&= \left(\frac{1}{2} \times 8\right) \times 10 \\
&= 40 \text{ cm}^2
\end{aligned}
$$

Step 2 : ***Multiply the area of the base by the height of the solid to find the volume***

$$
\begin{aligned}
\text{volume} &= \text{area of base} \times \text{height} \\
&= \frac{1}{2}b \times h \times H \\
&= 40 \times 20 \\
&= 800 \text{ cm}^3
\end{aligned}
$$

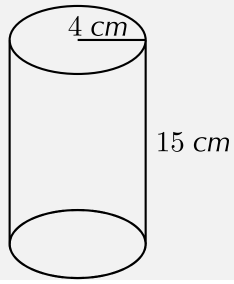
Step 3 : ***Write the final answer***

The volume of the triangular prism is 800 cm^3.

Example 7: *Finding the volume of a cylindrical prism*

QUESTION

Find the volume of the following cylinder (correct to 1 decimal place):



SOLUTION

Step 1 : ***Find the area of the base***

$$\begin{aligned} \text{area of circle} &= \pi r^2 \\ &= \pi(4)^2 \\ &= 16\pi \text{ cm}^2 \end{aligned}$$

Step 2 : ***Multiply the area of the base by the height of the solid to find the volume***

$$\begin{aligned} \text{volume} &= \text{area of base} \times \text{height} \\ &= \pi r^2 \times h \\ &= 16\pi \times 15 \\ &= 754{,}0 \text{ cm}^3 \end{aligned}$$

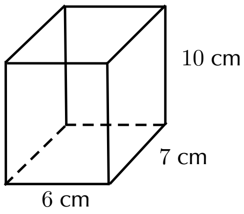
Step 3 : ***Write the final answer***

The volume of the cylinder is $754{,}0$ cm^3.

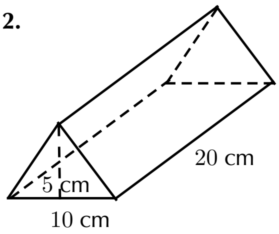
Exercise 12 - 3

Calculate the volumes of the following prisms (correct to 1 decimal place):

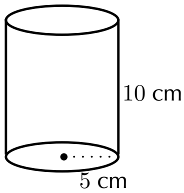
1.



2.



3.



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(1.) 00hn (2.) 00hp (3.) 00hq

12.3 Right pyramids, right cones and spheres

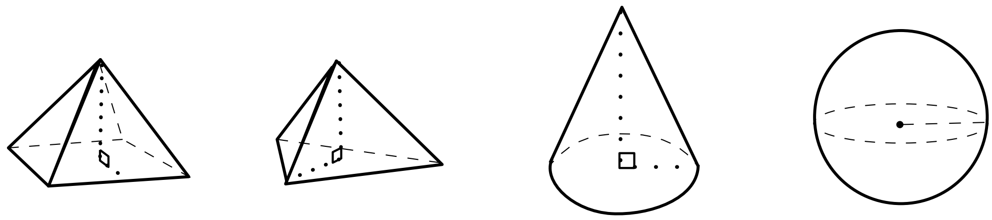
EMAER

DEFINITION: Pyramid

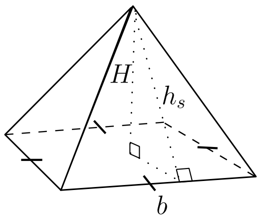
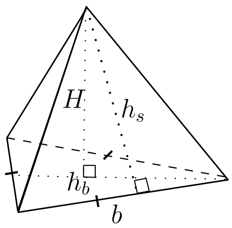
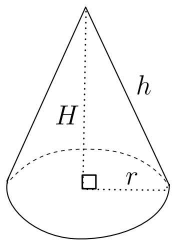
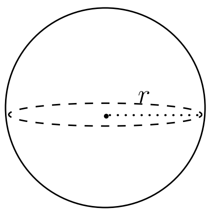
A pyramid is a geometric solid that has a polygon as its base and sides that converge at a point called the apex. In other words the sides are **not** perpendicular to the base.

The triangular pyramid and square pyramid take their names from the shape of their base. We call a pyramid a "right pyramid" if the line between the apex and the centre of the base is perpendicular to the base. Cones are similar to pyramids except that their bases are circles instead of polygons. Spheres are solids that are perfectly round and look the same from any direction.

Examples of a square pyramid, a triangular pyramid, a cone and a sphere:



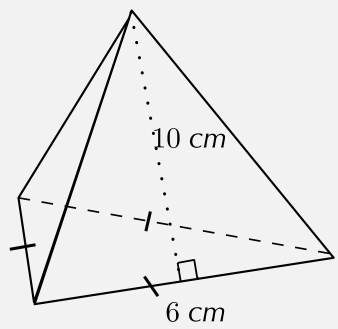
Surface area of pyramids, cones and spheres EMAES

Square pyramid		Surface area = area of base + area of triangular sides $= b^2 + 4\left(\frac{1}{2}bh_s\right)$ $= b(b + 2h_s)$
Triangular pyramid		Surface area = area of base + area of triangular sides $= \left(\frac{1}{2}b \times h_b\right) + 3\left(\frac{1}{2}b \times h_s\right)$ $= \frac{1}{2}b(h_b + 3h_s)$
Right cone		Surface area = area of base + area of walls $= \pi r^2 + \frac{1}{2} \times 2\pi rh$ $= \pi r(r + h)$
Sphere		Surface area $= 4\pi r^2$

Example 8: *Finding the surface area of a triangular pyramid*

QUESTION

Find the surface area of the following triangular pyramid (correct to one decimal place):

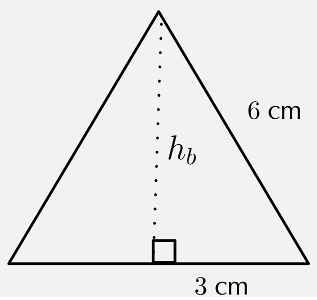


SOLUTION

Step 1 : ***Find the area of the base***

$$\text{area of base triangle} = \frac{1}{2}bh_b$$

To find the height of the base triangle (h_b) we use the Theorem of Pythagoras:



$$
\begin{aligned}
6^2 &= 3^2 + h_b^2 \\
\therefore h_b &= \sqrt{6^2 - 3^2} \\
&= 3\sqrt{3} \\
\therefore \text{ area of base triangle} &= \frac{1}{2} \times 6 \times 3\sqrt{3} \\
&= 9\sqrt{3} \text{ cm}^2
\end{aligned}
$$

Step 2 : ***Find the area of the sides***

$$
\begin{aligned}
\text{area of sides} &= 3\left(\frac{1}{2} \times b \times h_s\right) \\
&= 3\left(\frac{1}{2} \times 6 \times 10\right) \\
&= 90 \text{ cm}^2
\end{aligned}
$$

Step 3 : ***Find the sum of the areas***

$$9\sqrt{3} + 90 = 105{,}6 \text{ cm}^2$$

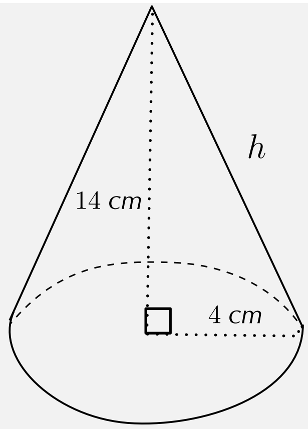
Step 4 : ***Write the final answer***

The surface area of the triangular pyramid is $105{,}6$ cm^2.

Example 9: *Finding the surface area of a cone*

QUESTION

Find the surface area of the following cone (correct to 1decimal place):



SOLUTION

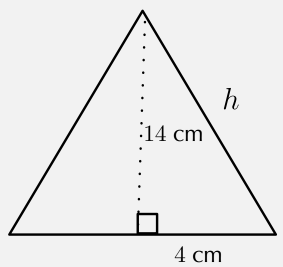
Step 1 : ***Find the area of the base***

$$\begin{aligned}\text{area of base circle} &= \pi r^2 \\ &= \pi \times 4^2 \\ &= 16\pi\end{aligned}$$

Step 2 : ***Find the area of the walls***

$$\text{area of sides} = \pi r h$$

To find the slant height h we use the Theorem of Pythagoras:



$$\begin{aligned}h^2 &= 4^2 + 14^2 \\ \therefore h &= \sqrt{4^2 + 14^2} \\ &= 2\sqrt{53}\text{ cm}\end{aligned}$$

$$\begin{aligned} \text{area of walls} &= \frac{1}{2}2\pi rh \\ &= \pi(4)(2\sqrt{53}) \\ &= 8\pi\sqrt{53}\ \text{cm}^2 \end{aligned}$$

Step 3 : ***Find the sum of the areas***

$$\begin{aligned} \text{total surface area} &= 16\pi + 8\pi\sqrt{53} \\ &= 233{,}2\ \text{cm}^2 \end{aligned}$$

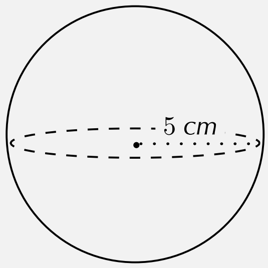
Step 4 : ***Write the final answer***

The surface area of the cone is $233{,}2$ cm^2.

Example 10: *Finding the surface area of a sphere*

QUESTION

Find the surface area of the following sphere (correct to 1 *decimal place):*



SOLUTION

$$\begin{aligned} \text{surface area of sphere} &= 4\pi r^2 \\ &= 4\pi 5^2 \\ &= 100\pi \\ &= 314{,}2\ \text{cm}^2 \end{aligned}$$

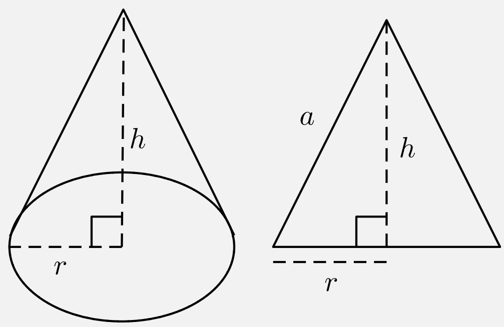
Example 11: *Examining the surface area of a cone*

QUESTION

If a cone has a height of h and a base of radius r, show that the surface area is: $\pi r^2 + \pi r\sqrt{r^2 + h^2}$.

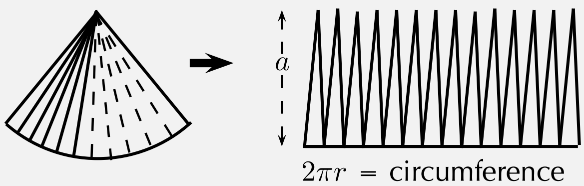
SOLUTION

Step 1 : ***Sketch and label the cone***



Step 2 : ***Identify the faces that make up the cone***

The cone has two faces: the base and the walls. The base is a circle of radius r and the walls can be opened out to a sector of a circle:



This curved surface can be cut into many thin triangles with height close to a (where a is the slant height). The area of these triangles or sectors can be summed as follows:

$$\begin{aligned}\text{Area of sector} &= \frac{1}{2} \times \text{base} \times \text{height (of a small triangle)}\\ &= \frac{1}{2} \times 2\pi r \times a\\ &= \pi r a\end{aligned}$$

Step 3 : ***Calculate** a*

a can be calculated using the Theorem of Pythagoras:

$$a = \sqrt{r^2 + h^2}$$

Step 4 : ***Calculate the area of the circular base** (A_b)*

$$A_b = \pi r^2$$

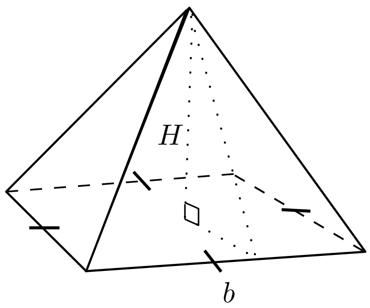
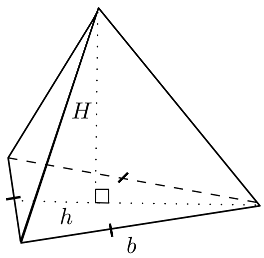
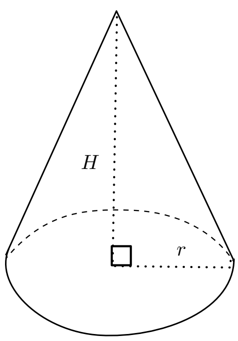
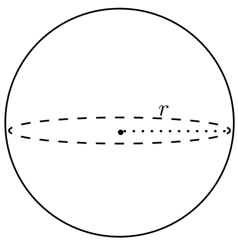
Step 5 : ***Calculate the area of the curved walls** (A_w)*

$$\begin{aligned} A_w &= \pi r a \\ &= \pi r \sqrt{r^2 + h^2} \end{aligned}$$

Step 6 : ***Find the sum of the areas** A*

$$\begin{aligned} A &= A_b + A_w \\ &= \pi r^2 + \pi r \sqrt{r^2 + h^2} \\ &= \pi r (r + \sqrt{r^2 + h^2}) \end{aligned}$$

Volume of pyramids, cones and spheres EMAET

Square pyramid		Volume $= \frac{1}{3} \times$ area of base $\times$ height of pyramid $= \frac{1}{3} \times b^2 \times H$
Triangular pyramid		Volume $= \frac{1}{3} \times$ area of base $\times$ height of pyramid $= \frac{1}{3} \times \frac{1}{2}bh \times H$
Right cone		Volume $= \frac{1}{3} \times$ area of base $\times$ height of cone $= \frac{1}{3} \times \pi r^2 \times H$
Sphere		Volume $= \frac{4}{3}\pi r^3$

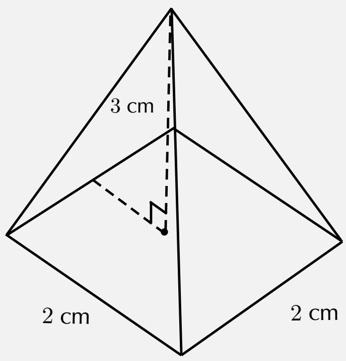
Example 12: *Finding the volume of a square pyramid*

QUESTION

Find the volume of a square pyramid with a height of 3 cm and a side length of 2 cm.

SOLUTION

Step 1 : ***Sketch and label the pyramid***



Step 2 : ***Select the correct formula and substitute the given values***

$$\text{volume} = \frac{1}{3} \times b^2 \times H$$

We are given $b = 2$ and $H = 3$, therefore

$$\begin{aligned} \text{volume} &= \frac{1}{3} \times 2^2 \times 3 \\ &= \frac{1}{3} \times 12 \\ &= 4 \text{ cm}^3 \end{aligned}$$

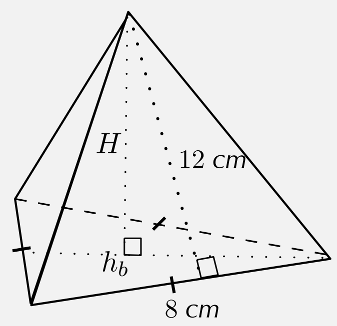
Step 3 : ***Write the final answer***

The volume of the square pyramid is 4 cm^3.

Example 13: *Finding the volume of a triangular pyramid*

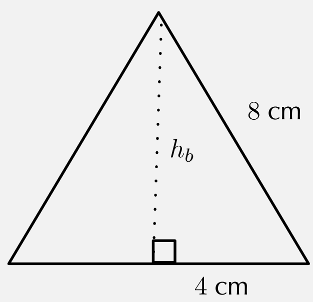
QUESTION

Find the volume of the following triangular pyramid (correct to 1 decimal place):



SOLUTION

Step 1 : ***Sketch the base triangle and calculate its area***

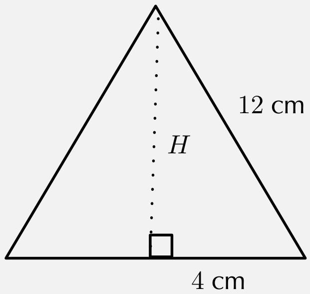


The height of the base triangle (h_b) can be calculated using the Theorem of Pythagoras:

$$
\begin{aligned}
8^2 &= 4^2 + h_b^2 \\
\therefore h_b &= \sqrt{8^2 - 4^2} \\
&= 4\sqrt{3} \text{ cm}
\end{aligned}
$$

$$\therefore \text{ area of base triangle} = \frac{1}{2}b \times h_b$$
$$= \frac{1}{2} \times 8 \times 4\sqrt{3}$$
$$= 16\sqrt{3} \text{ cm}^2$$

Step 2 : ***Sketch the side triangle and calculate pyramid height*** H



$$12^2 = 4^2 + H^2$$
$$\therefore H = \sqrt{12^2 - 4^2}$$
$$= 8\sqrt{2} \text{ cm}$$

Step 3 : ***Calculate the volume of the pyramid***

$$\text{volume} = \frac{1}{3} \times \frac{1}{2}bh_b \times H$$
$$= \frac{1}{3} \times 16\sqrt{3} \times 8\sqrt{2}$$
$$= 104{,}5 \text{ cm}^3$$

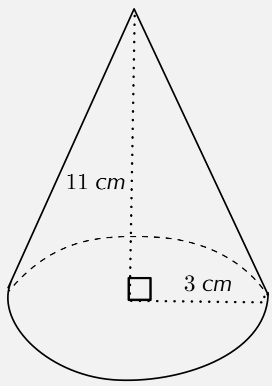
Step 4 : ***Write the final answer***

The volume of the triangular pyramid is $104{,}5$ cm^3.

Example 14: *Finding the volume of a cone*

QUESTION

Find the volume of the following cone (correct to 1 decimal place):



SOLUTION

Step 1 : ***Find the area of the base***

$$\begin{aligned}\text{area of circle} &= \pi r^2 \\ &= \pi \times 3^2 \\ &= 9\pi \text{ cm}^2\end{aligned}$$

Step 2 : ***Calculate the volume***

$$\begin{aligned}\text{volume} &= \frac{1}{3} \times \pi r^2 \times H \\ &= \frac{1}{3} \times 9\pi \times 11 \\ &= 103{,}7 \text{ cm}^3\end{aligned}$$

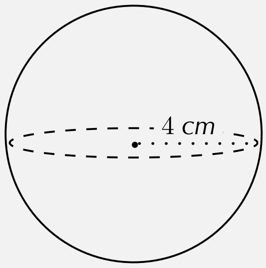
Step 3 : ***Write the final answer***

The volume of the cone is $103{,}7$ cm^3.

Example 15: *Finding the volume of a sphere*

QUESTION

Find the volume of the following sphere (correct to 1 decimal place):



SOLUTION

Step 1 : ***Use the formula to find the volume***

$$\begin{aligned} \text{volume} &= \frac{4}{3}\pi r^3 \\ &= \frac{4}{3}\pi(4)^3 \\ &= 268{,}1 \text{ cm}^3 \end{aligned}$$

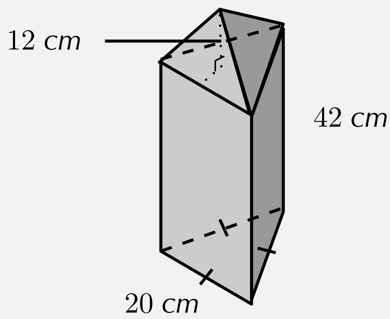
Step 2 : ***Write the final answer***

The volume of the sphere is $268{,}1$ cm^3.

Example 16: *Finding the volume of a complex object*

QUESTION

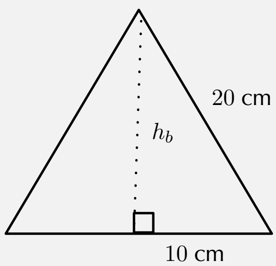
A triangular pyramid is placed on top of a triangular prism, as shown below. The base of the prism is an equilateral triangle of side length 20 *cm and the height of the prism is* 42 *cm. The pyramid has a height of* 12 *cm. Calculate the total volume of the object.*



SOLUTION

Step 1 : ***Calculate the volume of the prism***

(a) Find the height of the base triangle



$$\begin{aligned} 20^2 &= 10^2 + h_b^2 \\ \therefore h_b &= \sqrt{20^2 - 10^2} \\ &= 10\sqrt{3} \text{ cm} \end{aligned}$$

(b) Find the area of the base triangle

$$\begin{aligned} \text{area of base triangle} &= \frac{1}{2} \times 20 \times 10\sqrt{3} \\ &= 100\sqrt{3} \text{ cm}^2 \end{aligned}$$

(c) Find the volume of the prism

$$\begin{aligned}\therefore \text{volume of prism} &= \text{area of base triangle} \times \text{height of prism}\\ &= 100\sqrt{3} \times 42\\ &= 4\ 200\ \sqrt{3}\ \text{cm}^3\end{aligned}$$

Step 2 : ***Calculate the volume of the pyramid***

The area of the base triangle is equal to the area of the base of the pyramid.

$$\begin{aligned}\therefore \text{volume of pyramid} &= \frac{1}{3}(\text{area of base}) \times H\\ &= \frac{1}{3} \times 100\sqrt{3} \times 12\\ &= 400\sqrt{3}\ \text{cm}^3\end{aligned}$$

Step 3 : ***Calculate the total volume***

$$\begin{aligned}\text{total volume} &= 4\ 200\ \sqrt{3} + 400\sqrt{3}\\ &= 4\ 600\ \sqrt{3}\\ &= 7\ 967{,}4\ \text{cm}^3\end{aligned}$$

Therefore the total volume of the object is $7\ 967{,}4$ cm^3.

Example 17: *Finding the surface area of a complex object*

QUESTION

With the same complex object as in the previous Example, you are given the additional information that the slant height h_s of the triangular pyramid is $13{,}3$ cm. Now calculate the total surface area of the object.

SOLUTION

Step 1 : ***Calculate the surface area of each exposed face of the pyramid***

$$\begin{aligned}\text{area of one pyramid face} &= \frac{1}{2}b \times h_s \\ &= \frac{1}{2} \times 20 \times 13{,}3 \\ &= 133 \text{ cm}^2\end{aligned}$$

Because the base triangle is equilateral, each face has the same base, and therefore the same surface area. Therefore the surface area for each face of the pyramid is 133 cm^2.

Step 2 : ***Calculate the surface area of each side of the prism***

Each side of the prism is a rectangle with base $b = 20$ cm and height $h_p = 42$ cm.

$$\begin{aligned}\text{area of one prism side} &= b \times h_p \\ &= 20 \times 42 \\ &= 840 \text{ cm}^2\end{aligned}$$

Because the base triangle is equilateral, each side of the prism has the same area. Therefore the surface area for each side of the prism is 840 cm^2.

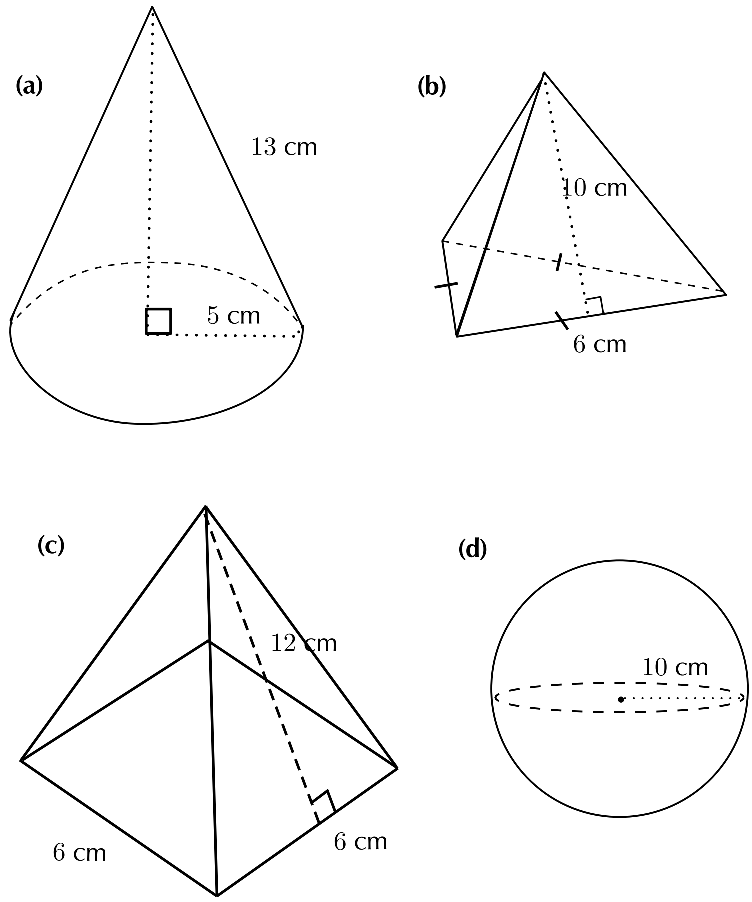
Step 3 : ***Calculate the total surface area of the object***

$$\begin{aligned}\text{total surface area} &= \text{area of base of prism} + \text{area of sides of prism} + \\ &\quad\ \text{area of exposed faces of pyramid} \\ &= (100\sqrt{3}) + 3(840) + 3(133) \\ &= 3\ 092{,}2 \text{ cm}^2\end{aligned}$$

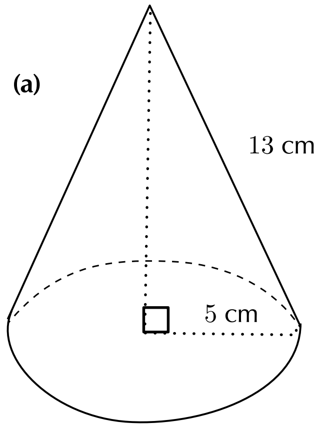
Therefore the total surface area (of the exposed faces) of the object is $3\ 092{,}2$ cm^2.

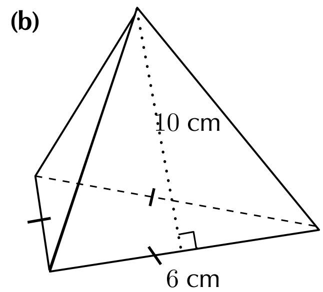
Exercise 12 - 4

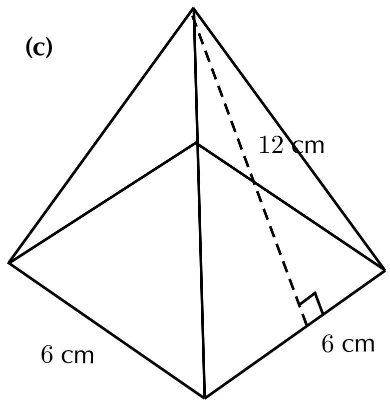
1. Find the total surface area of the following objects (correct to 1 decimal place if necessary):

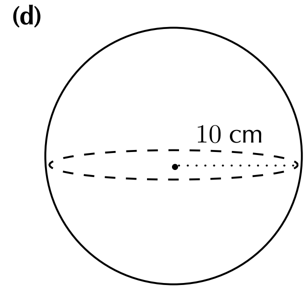


2. Find the volume of the following objects (round off to 1 decimal place if needed):

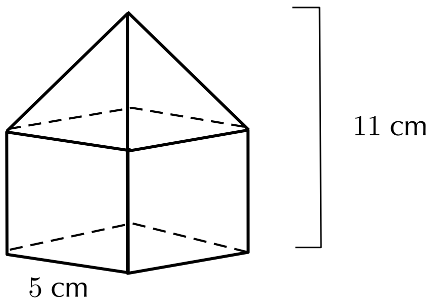








3. The solid below is made up of a cube and a square pyramid. Find it's volume and surface area (correct to 1 decimal place):



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(1.) 00hr (2.) 00hs (3.) 00ht

12.4 The effect of multiplying a dimension by a factor of k

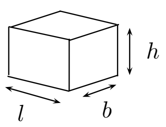
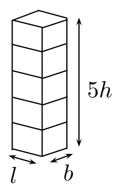
EMAEU

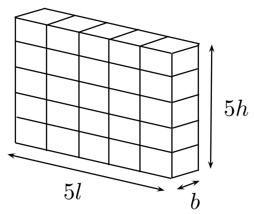
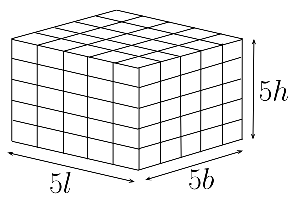
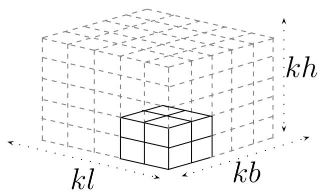
When one or more of the dimensions of a prism or cylinder is multiplied by a constant, the surface area and volume will change. The new surface area and volume can be calculated by using the formulae from the preceding section.

It is possible to see a relationship between the change in dimensions and the resulting change in surface area and volume. These relationships make it simpler to calculate the new volume or surface area of an object when its dimensions are scaled up or down.

Video: VMctl at www.everythingmaths.co.za

Consider a rectangular prism of dimensions l, b and h. Below we multiply one, two and three of its dimensions by a constant factor of 5 and calculate the new volume and surface area.

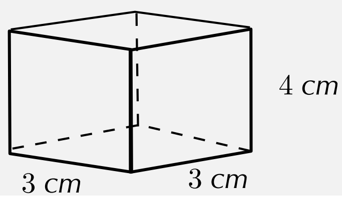
Dimensions	**Volume**	**Surface**
Original dimensions 	$V = l \times b \times h$ $= lbh$	$A = 2[(l \times h) + (l \times b) + (b \times h)]$ $= 2(lh + lb + bh)$
Multiply one dimension by 5 	$V_1 = l \times b \times 5h$ $= 5(lbh)$ $= 5V$	$A_1 = 2[(l \times 5h) + (l \times b) + (b \times 5h)]$ $= 2(5lh + lb + 5bh)$

Dimensions	**Volume**	**Surface**
Multiply two dimensions by 5 	$V_2 = 5l \times b \times 5h$ $= 5.5(lbh)$ $= 5^2 \times V$	$A_2 = 2[(5l \times 5h) + (5l \times b) + (b \times 5h)]$ $= 2 \times 5(5lh + lb + bh)$
Multiply all three dimensions by 5 	$V_3 = 5l \times 5b \times 5h$ $= 5^3(lbh)$ $= 5^3V$	$A_3 = 2[(5l \times 5h) + (5l \times 5b) + (5b \times 5h)]$ $= 2(5^2lh + 5^2lb + 5^2bh)$ $= 5^2 \times 2(lh + lb + bh)$ $= 5^2A$
Multiply all three dimensions by k 	$V_k = kl \times kb \times kh$ $= k^3(lbh)$ $= k^3V$	$A_k = 2[(kl \times kh) + (kl \times kb) + (kb \times kh)]$ $= k^2 \times 2(lh + lb + bh)$ $= k^2A$

Example 18: *Calculating the new dimensions of a rectangular prism*

QUESTION

Consider a square prism with a height of 4 cm and base lengths of 3 cm.



1. *Calculate the surface area and volume.*
2. *Calculate the new surface area (A_n) and volume (V_n) if the base lengths are multiplied by a constant factor of 3.*
3. *Express the new surface area and volume as a factor of the original surface area and volume.*

SOLUTION

Step 1 : **Calculate the original volume and surface area**

$$\begin{aligned} V &= l \times b \times h \\ &= 3 \times 3 \times 4 \\ &= 36 \text{ cm}^3 \end{aligned}$$

$$\begin{aligned} A &= 2[(l \times h) + (l \times b) + (b \times h)] \\ &= 2[(3 \times 4) + (3 \times 3) + (3 \times 4)] \\ &= 66 \text{ cm}^2 \end{aligned}$$

Step 2 : **Calculate the new volume and surface area**

Two of the dimensions are multiplied by a factor of 3.

$$\begin{aligned} V_n &= 3l \times 3b \times h \\ &= 3.3 \times 3.3 \times 4 \\ &= 324 \text{ cm}^3 \end{aligned}$$

$$\begin{aligned} A_n &= 2[(3l \times h) + (3l \times 3b) + (3b \times h)] \\ &= 2[(3.3 \times 4) + (3.3 \times 3.3) + (3.3 \times 4)] \\ &= 306 \text{ cm}^2 \end{aligned}$$

Step 3 : **Express the new dimensions as a factor of the original dimensions**

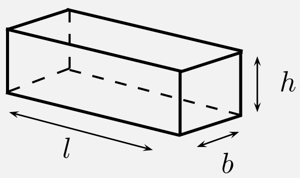
$$
\begin{aligned}
V &= 36 \\
V_n &= 324 \\
\frac{V_n}{V} &= \frac{324}{36} \\
&= 9 \\
\therefore V_n &= 9V \\
&= 3^2V
\end{aligned}
$$

$$
\begin{aligned}
A &= 66 \\
A_n &= 306 \\
\frac{A_n}{A} &= \frac{306}{66} \\
\therefore A_n &= \frac{306}{66}A \\
&= \frac{51}{11}A
\end{aligned}
$$

Example 19: *Multiplying the dimensions of a rectangular prism by k*

QUESTION

Prove that if the height of a rectangular prism with dimensions l, b and h is multiplied by a constant value of k, the volume will also increase by a factor k.



SOLUTION

Step 1 : ***Calculate the original volume***

We are given the original dimensions l, b and h and so the original volume is $V = l \times b \times h$.

Step 2 : ***Calculate the new volume***

The new dimensions are l, b, and kh and so the new volume is

$$\begin{aligned} V_n &= l \times b \times (kh) \\ &= k(lbh) \\ &= kV \end{aligned}$$

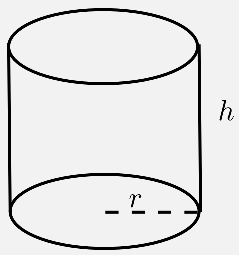
Step 3 : ***Write the final answer***

If the height of a rectangular prism is multiplied by a constant k, then the volume also increases by a factor of k.

Example 20: *Multiplying the dimensions of a cylinder by k*

QUESTION

Consider a cylinder with a radius of r and a height of h. Calculate the new volume and surface area (expressed in terms of r and h) if the radius is multiplied by a constant factor of k.



SOLUTION

Step 1 : ***Calculate the original volume and surface area***

$$V = \pi r^2 \times h$$
$$A = \pi r^2 + 2\pi r h$$

Step 2 : ***Calculate the new volume and surface area***

The new dimensions are kr and h.

$$\begin{aligned} V_n &= \pi(kr)^2 \times h \\ &= \pi k^2 r^2 \times h \\ &= k^2 \times \pi r^2 h \\ &= k^2 V \end{aligned}$$

$$\begin{aligned} A_n &= \pi(kr)^2 + 2\pi(kr)h \\ &= \pi k^2 r^2 + 2\pi k r h \\ &= k^2(\pi r^2) + k(2\pi r h) \end{aligned}$$

Exercise 12 - 5

1. If the height of a prism is doubled, how much will its volume increase?
2. Describe the change in the volume of a rectangular prism if:
   (a) length and breadth increase by a constant factor of 3.
   (b) length, breadth and height are multiplied by a constant factor of 2.
3. Given a prism with a volume of 493 cm^3 and a surface area of $6\ 007$ cm^2, find the new surface area and volume for a prism if all dimensions are increased by a constant factor of 4.

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(1.) 00hu (2.) 00hv (3.) 00hw

Chapter 12 | Summary

Summary presentation: VMdml at www.everythingmaths.co.za

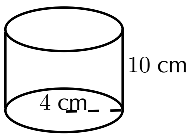
- Area is the two dimensional space inside the boundary of a flat object. It is measured in square units.
- Area formulae:
  - square: s^2
  - rectangle: $b \times h$
  - triangle: $\frac{1}{2}b \times h$
  - trapezium: $\frac{1}{2}(a + b) \times h$
  - parallelogram: $b \times h$
  - circle: πr^2
- Surface area is the total area of the exposed or outer surfaces of a prism.
- A net is the unfolded "plan" of a solid.
- Volume is the three dimensional space occupied by an object, or the contents of an object. It is measured in cubic units.
  - Volume of a rectangular prism: $l \times b \times h$
  - Volume of a triangular prism: $(\frac{1}{2}b \times h) \times H$
  - Volume of a square prism or cube: s^3
  - Volume of a cylinder: $\pi r^2 \times h$
- A pyramid is a geometric solid that has a polygon as its base and sides that converge at a point called the apex. The sides are not perpendicular to the base.
- Surface area formulae:
  - square pyramid: $b(b + 2h)$
  - triangular pyramid: $\frac{1}{2}b(h_b + 3h_s)$
  - right cone: $\pi r(r + h_s)$
  - sphere: $4\pi r^2$
- Volume formulae:
  - square pyramid: $\frac{1}{3} \times b^2 \times H$
  - triangular pyramid: $\frac{1}{3} \times \frac{1}{2}bh \times H$

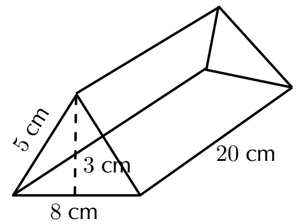
- right cone: $\frac{1}{3} \times \pi r^2 \times H$
- sphere: $\frac{4}{3}\pi r^3$

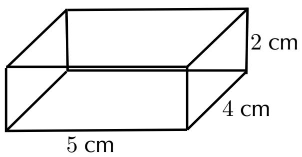
- Multiplying one or more dimensions of a prism or cylinder by a constant k affects the surface area and volume.

Chapter 12 End of Chapter Exercises

1. Consider the solids below and answer the questions that follow (correct to 1 decimal place, if necessary):

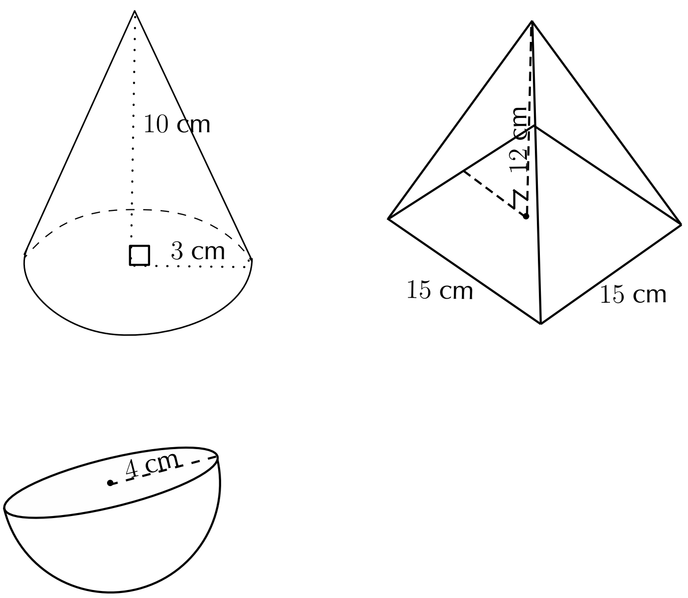






(a) Calculate the surface area of each solid.
(b) Calculate volume of each solid.
(c) If each dimension of the solids is increased by a factor of 3, calculate the new surface area of each solid.
(d) If each dimension of the solids is increased by a factor of 3, calculate the new volume of each solid.

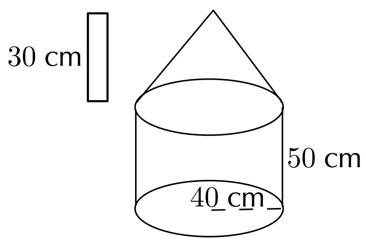
2. Consider the solids below:



(a) Calculate the surface area of each solid.

(b) Calculate the volume of each solid.

3. Calculate the volume and surface area of the solid below (correct to 1 decimal place):



More practice video solutions or help at www.everythingmaths.co.za

(1.) 00hx (2.) 00hy (3.) 00hz

Exercise Solutions 13

1. Algebraic Expressions EMAEV

Exercise 1-1

1. (a) Rational, integer
   (b) Irrational
   (c) Rational, integer, whole, natural number
   (d) Irrational
2. (a) Rational
   (b) Rational
   (c) Rational
   (d) Irrational
3. (a) Rational
   (b) Rational
   (c) Irrational
   (d) Rational
4. (a) $\frac{1}{10}$
   (b) $\frac{3}{25}$
   (c) $\frac{29}{50}$
   (d) $\frac{2589}{10000}$
5. (a) $0{,}\overline{1}$
   (b) $0{,}\overline{12}$
   (c) $0{,}\overline{123}$
   (d) $0{,}11\overline{4145}$
6. (a) $0{,}\overline{6}$
   (b) $1{,}\overline{27}$
   (c) $4{,}8\overline{3}$
   (d) $2{,}\overline{1}$
7. (a) $\frac{5}{9}$
   (b) $\frac{57}{90}$
   (c) $\frac{526}{99}$

Exercise 1-2

1. $12{,}567$
2. $3{,}317$
3. $0{,}267$
4. $1{,}913$
5. $6{,}325$
6. $0{,}056$

Exercise 1-3

1. 4 and 5
2. 5 and 6
3. 1 and 2
4. 4 and 5

Exercise 1-4

1. $2y^2 + 8y$
2. $y^2 + 7y + 10$
3. $2 + 3t + 2t^2$
4. $x^2 - 16$
5. $6p^2 + 29p + 9$
6. $3k^2 + 16k - 12$
7. $s^2 + 12s + 36$
8. $-49 + x^2$
9. $9x^2 - 1$
10. $14k^2 + 17k + 6$
11. $1 - 8x + 16x^2$
12. $y^2 - 2y - 15$
13. $16 - x^2$
14. $81 + 18x + x^2$
15. $-10y^3 + 4y^2 + 103y - 132$
16. $-14y^3 + 26y^2 + 4y - 16$
17. $-20y^3 - 116y^2 - 13y + 6$
18. $-144y^3 + 96y^2 - 3y - 9$
19. $-20y^2 - 80y - 30$
20. $-10y^7 - 39y^6 - 36y^5$
21. $84y^2 - 153y + 33$
22. $49y^3 + 42y^2 + 79y + 10$
23. $72y^2 - 18y + 27$
24. $-6y^6 - 85y^4 + 3y^3 + 176y^2 + 48y$

Exercise 1-5

1. $2(l + w)$
2. $4(3x + 8y)$
3. $2x(3x + 1 + 5x^2)$
4. $xy(2y^2 + yz + 3)$
5. $-2ab(b + a)$
6. $7a + 4$
7. $10(2a - 1)$
8. $3b(6a - c)$
9. $6k(2j + 3q)$
10. $(4k - 2)(4k + 2)$
11. $3(a^2 + 2a - 9)$
12. $12a(2a^2 - 1)$
13. $-2a(b + 4)$
14. $8kj(3 - 2k)$
15. $-ab(a + b)$
16. $12jk^2(2j + 1)$
17. $18b^2q(4 - bq)$
18. $(3 - y)(-4 + k)$
19. $(a - 1)(a - 5)(a + 5)$
20. $(b + 4)(m)(b - 6)$
21. $(a + 7)(a^2 + 9)$
22. $(b - 4)(3b + 7)$
23. $(abc - 1)(abc + 1)$

Exercise 1-6

1. $(3 + a)(2x + 1)$
2. $(x + 5)(x - 6)$

3. $(5-a)(x+2y)$
4. $(a-x)(a-2)$
5. $(y+2)(5x-3)$
6. $(-a+b)(a+1)$

Exercise 1-7

1. (a) $(x+5)(x+3)$
   (b) $(x+6)(x+4)$
   (c) $(x+8)(x+1)$
   (d) $(x+7)(x+2)$
   (e) $(x+12)(x+3)$
   (f) $(x+6)(x+6)$
2. (a) $(x+5)(x-3)$
   (b) $(x+3)(x-1)$
   (c) $(x+4)(x-2)$
   (d) $(x+5)(x-4)$
   (e) $(x-5)(x+4)$
   (f) $2(x+10)(x+1)$
3. (a) $(3x+1)(x+6)$
   (b) $(6x+1)(x+1)$
   (c) $(6x+1)(2x+1)$
   (d) $(2x+1)(4x+1)$
4. (a) $(3x+1)(x+6)$
   (b) $(7x+1)(x-1)$
   (c) $(4x-1)(2x-1)$
   (d) $(6x+3)(x-3)$

Exercise 1-8

1. $(x+2)(x^2-2x+4)$
2. $(3-m)(9+3m+m^2)$
3. $2(x-y)(x^2+xy+y^2)$
4. $3(k+3q)(k^2-3kq+9q^2)$
5. $(4t-1)(16t^2+4t+1)$
6. $(8x-1)(8x+1)$
7. $(5x+1)(25x^2-5x+1)$
8. No solution
9. $z(1-5z)(1+5z+25z^2)$
10. $(2m^2+n^3)(4m^4-2m^2n^3+n^6)$
11. $(p^5-\frac{1}{2}y^4)(p^10+\frac{1}{2}p^5y^4+\frac{1}{4}y^8$
12. $[1-x+y][1+x-y+x^2-2xy+y^2]$

Exercise 1-9

1. $\frac{a}{5}$
2. $\frac{a+5}{2}$
3. 5
4. a
5. $\frac{3a}{2}$
6. $\frac{a+3}{a+2}$
7. $\frac{a(3b+1)}{b}$
8. $\frac{4xy}{3}$
9. $\frac{p(y-2)}{3y}$
10. $\frac{3(a+3)}{98}$
11. $\frac{4a^2(a-5)}{6(a+5)^2}$
12. $\frac{(3x+4)^2}{96p^2}$
13. $\frac{4}{3}$
14. $2a$
15. $\frac{3q}{8}$
16. $\frac{5(a+b)}{24b}$
17. af
18. $\frac{2z+4y+3x}{xyz}$
19. 5
20. $\frac{2(k-1)}{(k^2+2)(k+2)}$
21. $\frac{5t+7}{6q}$
22. $\frac{(5p-2)}{(p-2)^2(p+2)}$
23. $\frac{-xy}{x^2-y^2}$
24. $\frac{m+1}{m^2-mn+n^2}$
25. $\frac{f}{h^3-f^3}$
26. $\frac{2x-1}{6}$
27. 0
28. $-\frac{x^2-3x-1}{(x-1)^2(x^2+x+1)}$
29. $\frac{3(p^2-pq+q^2)}{p^2}$
30. $\frac{a^2+4b-4b^2}{(a-2b^2)(a+2b)}$

End of chapter exercises

1. (a) Rational
   (b) Irrational
   (c) Irrational
   (d) Irrational
2. (a) $\frac{3}{5}$
   (b) $\frac{3}{500}$
   (c) $1\frac{59}{100}$
   (d) $\frac{221}{18}$
4. $\frac{78}{90}$
5. (a) 0,5
   (b) 1,00
   (c) 0,11
   (d) 1,00
6. (a) 3,142
   (b) 1,618
   (c) 1,414
   (d) 2,718
9. (a) 3,142 $\frac{1\,571}{500}$
   (b) 1,618 $\frac{809}{500}$
   (c) 1,414 $\frac{707}{500}$
   (d) 2,718 $\frac{1\,359}{500}$
10. (a) 2 and 3
   (b) 3 and 4
   (c) 4 and 5
   (d) 5 and 6
   (e) 1 and 2
   (f) 2 and 3
   (g) 2 and 3
   (h) 3 and 4
11. 2 and 3
12. 3 and 4
13. (a) $(a-3)(a+3)$
   (b) $(m+6)(m-6)$
   (c) $(3b-9)(3b+9)$
   (d) $(4b+5a)(4b-5a)$
   (e) $(m+\frac{1}{3})(m-\frac{1}{3})$
   (f) $5(1-ab^3)(1+ab^3)$
   (g) $b(4a^2-9)(2a+3)(2a-3)$
   (h) $(a-5)(a-5)$
   (i) $(4b+7)(4b+7)$
   (j) $(2a-6b)(a-3b)$
   (k) $-4b^2(6b^3-1)(6b^3-1)$
   (l) $(2-x)(2+x)(4+x^2)$
   (m) $7(x-2)(x+y)$
   (n) $(y-10)(y+3)$
   (o) $(1-x)^2(1+x)$
   (p) $(p-1)(-3p-2)$
   (q) $(x+y)(1-x^2+xy-y^2)$
   (r) $x(x-2)+(1+y)(1-y)(1+y^2)$
   (s) $(x-1)(x^2+x+1)(4b-x)$
   (t) $3(p-\frac{1}{3})(p^2+\frac{p}{3}+\frac{1}{9})$
   (u) $(2x^2-5y^3)(4x^4+10x^2y^3+25y^6)$
   (v) $(-p)(12+18p+7p^2)$
14. (a) $-8a+4$
   (b) $125a^3-64b^3$

(c) $16m^4 - 81$
(d) $a^2 + 4ab + 4b^2 - c^2$
(e) $p^2 - 2pq + q^2$
(f) $\frac{12-x^2}{6x}$
(g) $\frac{-14}{(a+7)(a-7)}$
(h) $\frac{32x^3+x+2}{2x^3}$
(i) $\frac{4a-1}{(2a+1)(2a-1)(a-1)}$
(j) $\frac{x(x+1)}{x^2+x+6}$

15. $(3x-4)(x+2)$
16. $5x$
17. $8{,}85$
18. $a^2 + 2ab + 4b^2$
19. $9x^2 - 3x + 1$

2. Equations and inequalities

EMAEW

Exercise 2-1

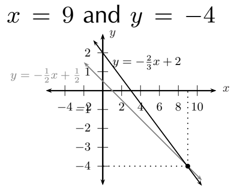
1. 5
2. 0
3. $\frac{-7}{8}$
4. 12
5. 11
6. $\frac{117}{20}$
7. 15
8. $\frac{9}{10}$
9. $\frac{100}{26}$
10. 1
11. 3
12. 6
13. $\frac{17}{12}$
14. $\frac{20}{3}$
15. 26
16. 5
17. 4
18. $-\frac{36}{11}$
19. 2
20. No solution
21. 5
22. 24
23. -6
24. $\frac{-7}{3}$
25. 2
26. $-\frac{17}{13}$
27. $-12{,}5$
28. $-\frac{17}{3}$
29. $-\frac{1}{6}$
30. -3

Exercise 2-2

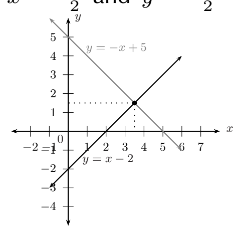
1. $\frac{-2}{3}$ or $\frac{4}{3}$
2. $\frac{9}{5}$ or -6
3. $\frac{-3}{2}$ or $\frac{3}{2}$
4. $-\frac{1}{2}$ or $\frac{9}{2}$
5. $\frac{3}{2}$
6. 0 or $\frac{-4}{5}$
7. 4 or $\frac{-3}{2}$
8. $\frac{6}{5}$ or $\frac{8}{3}$
9. 4 or 11
10. 2
11. $\frac{3}{5}$ or 3
12. 0 or 3
13. 5
14. $\sqrt{18}$ or $-\sqrt{18}$
15. -1 or 7
16. 7 or $-\frac{11}{4}$
17. $\frac{3}{7}$ or $-\frac{1}{2}$
18. -2 or 3
19. 1 or $-5\frac{3}{8}$
20. 0 or $\frac{2}{3}$

Exercise 2-3

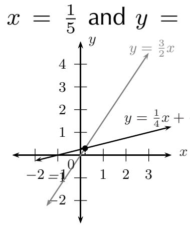
1. (a) $y = \frac{-3}{2}$ and $x = -7$
   (b) $y = 3$ and $x = 5$
   (c) $x = -1$ and $y = -1$
   (d) $a = 5\frac{1}{3}$ and $b = 1\frac{1}{3}$
   (e) $x = \frac{1}{7}$ and $y = -\frac{1}{4}$
2. (a) $x = 9$ and $y = -4$



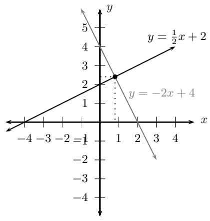
   (b) $x = \frac{7}{2}$ and $y = \frac{3}{2}$



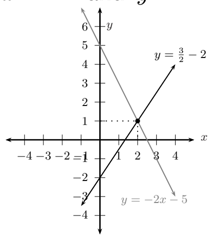
   (c) $x = \frac{1}{5}$ and $y = \frac{3}{10}$



   (d) $x = \frac{4}{5}$ and $y = 2\frac{2}{5}$



(e) $x = 2$ and $y = 1$



Exercise 2-4

1. $t = 2$ hrs
2. $L = 18$ and $S = 2$
3. $w = 28$, $l = 45$ and $d = 53$
4. -34
5. $30°$ and $60°$
6. $l = 16$ cm and $b = 8$ cm
7. $x = 7$ or $x = -3$
8. $l = 6$ cm and $w = 4$ cm
9. $\frac{19}{50}$
10. 9 and 11
11. $\frac{1}{2}$
12. 8 years old
13. Murunwa is 7 and Tshamano is 35

Exercise 2-5

1. $a = \frac{2(s-ut)}{t^2}$
2. $n = \frac{pV}{RT}$
3. $x = \frac{2b^2}{2b-1}$
4. $r = \sqrt{V}\pi h$
5. $h = \frac{E\lambda}{C}$
6. $h = \frac{A-2\pi r}{2\pi r}$
7. $\lambda = \frac{D}{ft}$
8. $m = \frac{E}{gh+\frac{1}{2}V^2}$
9. $x = -a$ or $x = -b$
10. $b = \pm\sqrt{c^2 - a^2}$
11. $u = \frac{vw}{w-v}$
12. $r = \pm\sqrt{R^2 - \frac{A}{\pi}}$
13. $C = \frac{160}{9} - \frac{5}{9}F$
14. $r = sqrt[3]\frac{3V}{4\pi}$

Exercise 2-6

1. $x < -2$
2. $x \geq -3$
3. $x < \frac{-5}{4}$
4. $x > \frac{2}{5}$
5. 0
6. $-1 \leq x < 4$
7. $-1 < x \leq 5$
8. $x > 2$
9. $x \geq 1$
10. $-12 \leq x \leq 1$
11. $y < \frac{6}{7}$
12. $-4 \leq y \leq 0$
13. $-20 < x < 7$

End of chapter exercises

1. **(a)** -8
   (b) $\frac{1}{2}$
   (c) No solution
   (d) -2
   (e) 1
   (f) 10
   (g) 5
   (h) $\frac{1}{3}$
   (i) $x = 2$ or $x = 1$
   (j) $y = 2$ or $y = -3$
   (k) $x = 2$ or $x = \frac{-9}{2}$
   (l) 4
   (m) $x \geq 2$
   (n) $x < -1$
   (o) $x > \frac{588}{10}$
   (p) $a \leq -\frac{7}{5}$
   (q) $-3 \leq k < 2$
   (r) $x = 7$ or $x = 6$
2. **(a)** $I = \frac{P}{V}$
   (b) $m = \frac{E}{c^2}$
   (c) $t = \frac{v-u}{a}$
   (d) $t = \frac{uv}{u+v}$
   (e) $C = \frac{5}{9}(F - 32)$
   (f) $y = mx + c$
3. **(a)** $x = 1$ and $y = 2$
   (b) $x = 4$ and $y = 2$
   (c) $x = 5$ and $y = -2$
   (d) No solution
4. **(a)** $x = \frac{120}{13}$
   (b) Ruler: R 5; pen: R 3
   (c) $\frac{5}{9}$ km
   (d) 8 km and 12 km
   (e) R 20,00

3. Exponentials

EMAEX

Exercise 3-1

1. 1
2. 16
3. $\frac{1}{36}$
4. 40
5. $\frac{27}{8}$
6. x^{3t+3}
7. 3^{2a+3}

8. a^{2x}
9. $\frac{8}{p^6}$
10. $8t^{12}$
11. 3^{2n+6}
12. $\frac{1}{27}$

Exercise 3-2

1. $3t^2$
2. $8x$
3. $\frac{1}{2}$
4. $\frac{1}{3}$
5. $3p^3$

Exercise 3-3

1. (a) 0
   (b) $-\frac{5}{2}$
   (c) -7
   (d) $\frac{5}{9}$
   (e) 4
   (f) $-\frac{1}{2}$
   (g) $\frac{4}{5}$
   (h) 3
   (i) 1
   (j) $\frac{3}{2}$ or 0
   (k) 1
   (l) $-3\frac{1}{2}$
   (m) -3
   (n) -3
   (o) -1 or 3
2. 7

End of chapter exercises

1. (a) $2t^3$
   (b) $5^{5x+y+3z}$
   (c) b^{k^2+k}
   (d) 864^p
   (e) $27m^t$
   (f) $\frac{1}{3x^5}$
   (g) $\frac{1}{625}$
   (h) $\frac{27}{4}$
   (i) $\frac{1}{27}$
   (j) $8x^{6a}y^{3b}$
   (k) 4^{x+3}
   (l) 22
   (m) -27
   (n) $\frac{6}{5}$
   (o) $\frac{1}{3}$
   (p) $\frac{1}{8}$
   (q) 14
   (r) $2^{2p} - 2^p + 1$
2. (a) -3
   (b) 1
   (c) $\frac{2}{3}$
   (d) $\frac{1}{2}$
   (e) -1
   (f) 16
   (g) $\frac{1}{16}$
   (h) -1
   (i) 1 or 16
   (j) 2
   (k) $\frac{1}{81}$
   (l) 81

4. Number patterns

EMAEY

Exercise 4-1

1. (a) $35; 45; 55$
   (b) $7; 12; 17$
   (c) $21; 18; 15$
2. (a) 8
   (b) -1
   (c) -9 and -5
3. (a) $T_n = 3n - 1$; $T_{10} = 29$; $T_{50} = 149$; $T_{100} = 299$
   (b) $T_n = 4n - 4$; $T_{10} = 36$; $T_{50} = 196$; $T_{100} = 396$
   (c) $T_n = 5 - 3n$; $T_{10} = -25$; $T_{50} = -145$; $T_{100} = -295$

End of chapter exercises

1. (a) 49
   (b) -10
   (c) $18{,}9$
2. (a) $T_n = 4n - 1$
   (b) $T_n = 3n - 5$
   (c) $T_n = 4n + 7$
   (d) $T_n = \frac{n}{3}$
3. 111
4. (a) 4
   (b) 3
   (c) $T_n = 3n + 1$
   (d) 76
5. 10
6. 77
7. (c) $9 + (10x + y) = 10(x + 1) + (y - 1)$

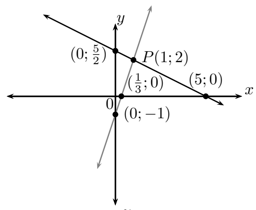
5. Functions

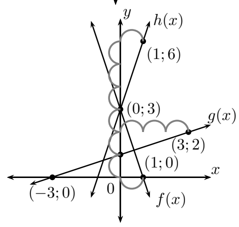
EMAEZ

Exercise 5-1

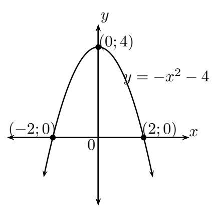
1. (a) $\{x : x \in \mathbb{R}, x \leq 7\}$
   (b) $\{y : y \in \mathbb{R}, -13 \leq y < 4\}$
   (c) $\{z : z \in \mathbb{R}, z > 35\}$
   (d) $\{t : t \in \mathbb{R}, \frac{3}{4} \leq t < 21\}$
   (e) $\{p : p \in \mathbb{R}, -\frac{1}{2} \leq p \leq \frac{1}{2}\}$
   (f) $\{m : m \in \mathbb{R}, m > -\sqrt{3}\}$
2. (a) $(-\infty; 6]$
   (b) $(-5; 5)$
   (c) $(\frac{1}{5}; \infty)$
   (d) $[21; 41)$

Exercise 5-2

1. (a) $(0; 1)$ and $(-1; 0)$; increasing
   (b) $(0; -1)$ and $(1; 0)$; increasing
   (c) $(0; -1)$ and $(\frac{1}{2}; 0)$; increasing
   (d) $(0; 1)$ and $(\frac{1}{3}; 0)$; decreasing
   (e) $(0; 2)$ and $(-3; 0)$; increasing
   (f) $(0; 3)$; horizontal line
   (g) $(0; 0)$; increasing
   (h) $(0; 3)$ and $(2; 0)$; decreasing
2. (a) $a(x) = -\frac{3}{4}x + 3$
   (b) $b(x) = \frac{3}{2}x - 6$
   (c) $p(x) = 3$
   (d) $d(x) = -\frac{3}{4}x$
3. 

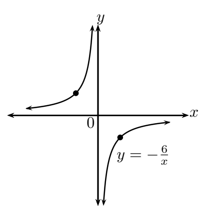
4. 

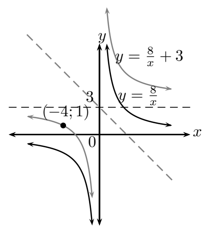
Exercise 5-3

2. 

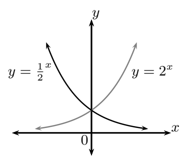
3. (a) $a = 1$ and $p = -9$
   (b) $b = -1$ and $q = 23$
   (c) $x \leq -4$ or $x \geq 4$
   (d) $x \geq 0$

Exercise 5-4

1. 

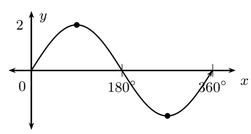
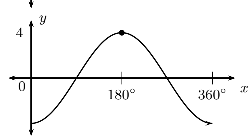
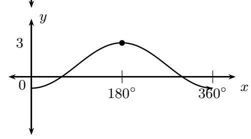
   (a) Yes
   (b) $y = -24$
   (c) Decrease
   (d) $y = 0$ and $x = 0$
   (e) $(-3; 2)$
2. 

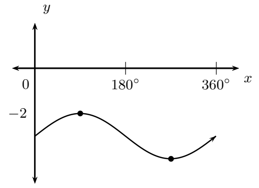
Exercise 5-5

1. 

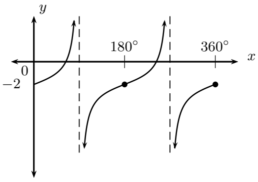
   (a) Asymptote
   (b) $y = \frac{1}{2}^x$
   (c) $(0; 1)$
2. (a) $f(x) = 3^x$
   (b) $h(x) = -3^x$
   (c) Range: $(-\infty; 0)$
   (d) $g(x) = 3^{-x}$
   (e) $j(x) = 2.3^x$
   (f) $k(x) = 3^x - 3$

Exercise 5-6

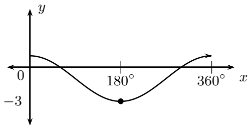
1. (a) 
   (b) 
   (c) 



(d)



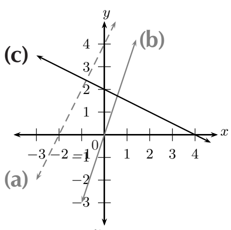
(e)

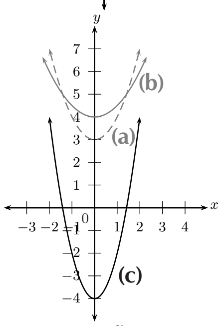


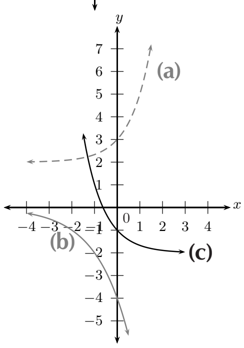
(f)

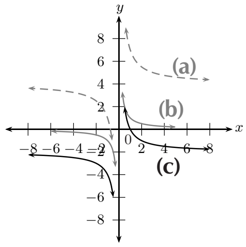
2. (a) $y = 2\ cos\ \theta$
   (b) $y = sin\ \theta + 1$

End of chapter exercises

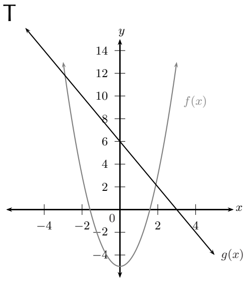
1. 

2. 

3. 

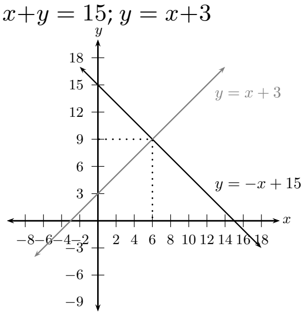
4. 

5. (a) F
   (b) F
   (c) T
   (d) F
   (e) F
   (f) T

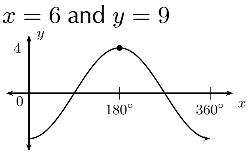
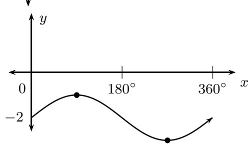
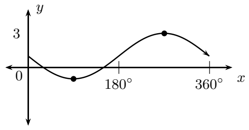
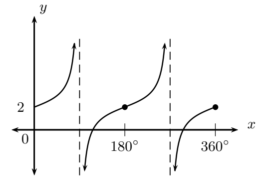
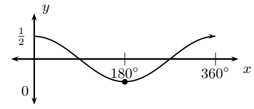
6. (a) 

   (b) $(-3; 12)$ and $(2; 2)$
   (c) i. $x \in (-\infty; \sqrt{3}) \cup (\sqrt{3}; \infty)$
   ii. $x \in (3; \infty)$
   iii. $x \in [-3; 2]$
   (d) $y = -2x^2 + 6$

7. 1,6 units

8. (a) $x+y = 15$; $y = x+3$
   (b) 

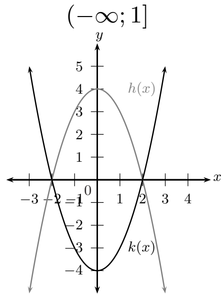
   (c) $x = 6$ and $y = 9$

9. (a) 
   (b) 
   (c) 
   (d) 
   (e) 

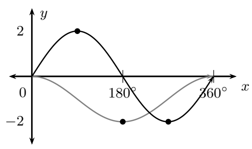
10. (a) $y = 3x$
    (b) $y = -2x^2 + 3$
    (c) $y = \frac{-3}{x}$
    (d) $y = x + 2$
    (e) $y = 5\ sin\ x + 1$
    (f) $y = 2.2^x + 1$
    (g) $y = -tan\ x - 2$

11. (a) $M(0; 1), N(0; -1)$
    (b) $MN = 2$ units
    (c) $PQ = 1$ unit
    (d) $y = 2^{-x}$
    (e) Range $y = 2^x$: $(0; \infty)$;
    Range $y = -2^x$: $(-\infty; 0)$

12. (a) $q = 1$
    (b) 7 units.
    (c) $y = -4^x$
    (d) $y = 4^x + 1$
    (e) $x = 0$
    (f) Range $f(x)$: $(0; \infty)$
    Range $g(x)$: $(-\infty; 1]$

13. 

    (b) $y = x^2 + 4$
    (c) Domain h: $(-\infty; \infty)$
    Range h: $[-4; \infty)$

14. 
    (a) $f(180°) = 0$
    (b) $g(180°) = -2$

(c) $g(270°)-f(270°) = 1$
(d) Domain: $[0°; 360°]$
Range: $[-2; 0]$
(e) Amplitude $= 2$;
Period $= 360°$

15. (a) $A(\sqrt{8}; \sqrt{8})$ and $B(-\sqrt{8}; -\sqrt{8})$
(b) $CD = 2\sqrt{8}$ units
(c) $AB = 8$ units
(d) $EF = 2$ units

16. (a) $A(-1; 0), B(1; 0)$ and $C(0; 3)$
(b) The two graphs intersect in the fourth quadrant.
(c) $D(2; -9)$
(d) $y = -6x + 3$

17. (a) Domain: $\{\theta : 0° \leq \theta \leq 360°, \theta \neq 90°; 270°\}$
(b) Amplitude $= 3$
(c) i. $\{0°; 180°; 360°\}$
ii. $(0°; 90°) \cup (270°; 360°)$
iii. $\{\theta : 90° < \theta < 270°, \theta \neq 180°\}$
iv. $(0°; 90°) \cup (270°; 360°)$

6. Finance and growth

EMAFA

Exercise 6-1

1. R 4 025
2. (a) R 324
(b) R 3 937,50
3. 19 years
4. 1,25% p.a

Exercise 6-2

1. (a) R 3 825
(b) R 4 743
(c) R 197,63
(d) R 5 418
2. (a) R 12 962,50
(b) R 4 462,50
(c) R 360,07
3. (a) R 5 400
(b) R 4 251,97

Exercise 6-3

1. R 4 044,69
2. 8,45% p.a
3. R 59 345,13

Exercise 6-4

1. R 2 174,77
2. R 38,64
3. 553 babies

Exercise 6-5

1. (a) R 1 400
(b) R 200
(c) R 100
2. (a) USA
(b) Sollie

End of chapter exercises

1. R 1 840
2. (a) R 534,25
(b) R 520
3. Bank B
4. R 200
5. R 200
6. (a) Simple interest
(b) Compound interest
7. (a) R 205
(b) R 286,52
(c) R 128
8. 1 AUD = 82,03 Yen
9. 8,5% p.a
10. (a) 62,3 million people
(b) $\approx$ 1,7%

7. Trigonometry

EMAFB

Exercise 7-1

1. (a) a =adj; b =hyp; c =opp
   (b) a =opp; b =adj; c =hyp
   (c) a =hyp; b =opp; c =adj
   (d) a =opp; b =hyp; c =adj
   (e) a =adj; b =hyp; c =opp
   (f) a =adj; b =opp; c =hyp
2. (a) $2{,}14$
   (b) $0{,}62$
   (c) $0{,}28$
   (d) $0{,}21$
   (e) $0{,}90$
   (f) $1{,}15$
   (g) $0{,}23$
   (h) $2{,}52$
   (i) $0{,}67$
3. (a) T
   (b) F
   (c) T
   (d) F
4. (a) $sin\ \hat{A} = \frac{CB}{AC}$
   (b) $cos\ \hat{A} = \frac{AB}{AC}$
   (c) $tan\ \hat{A} = \frac{CB}{AB}$
   (d) $sin\ \hat{C} = \frac{AB}{AC}$
   (e) $cos\ \hat{C} = \frac{CB}{AC}$
   (f) $tan\ \hat{C} = \frac{AB}{CB}$
5. (a) $\frac{\sqrt{3}}{2}$
   (b) $\frac{1}{2}$
   (c) $\sqrt{3}$
   (d) $\frac{1}{2}$
   (e) $\frac{\sqrt{3}}{2}$
   (f) $\frac{1}{\sqrt{3}}$
6. (a) $\frac{1}{\sqrt{2}}$
   (b) $\frac{1}{\sqrt{2}}$
   (c) 1

Exercise 7-2

1. (a) $\frac{1}{2}$
   (b) $1\frac{1}{2}$
   (c) $\frac{\sqrt{3}-1}{2}$

Exercise 7-3

1. (a) $a = 37{,}31$ units
   (b) $b = 8{,}91$ units
   (c) $c = 10{,}90$ units
   (d) $d = 21{,}65$ units
   (e) $e = 41{,}04$ units
   (f) $f = 33{,}43$ units
   (g) $g = 29{,}46$ units
   (h) $h = 10{,}00$ units
2. (a) $sin\ \hat{B} = \frac{AC}{AB} = \frac{AD}{BD}$
   (b) $cos\ \hat{D} = \frac{AD}{BD} = \frac{CD}{AD}$
   (c) $tan\ \hat{B} = \frac{AC}{BC} = \frac{AD}{AB}$
3. $MN = 12{,}86$ units; $NP = 15{,}32$ units

Exercise 7-4

1. (a) $59{,}5°$
   (b) $53{,}1°$
   (c) $71{,}3°$
   (d) $80{,}1°$
   (e) $41{,}8°$
   (f) No solution
   (g) $41{,}4°$
   (h) $18{,}1°$
   (i) $40{,}5°$
   (j) $53{,}1°$
   (k) $18{,}4°$
   (l) No solution
   (m) $109{,}8°$
   (n) $26{,}6°$
   (o) $17{,}7°$
2. (a) $24{,}0°$
   (b) $35{,}2°$
   (c) $50{,}6°$
   (d) $26{,}3°$
   (e) $37{,}0°$
   (f) $45{,}0°$

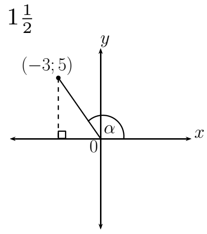
Exercise 7-5

1. $53{,}13°$
2. $35{,}30°$
3. 26 m
4. $15{,}05$ m

Exercise 7-6

1. (a) $OB = \sqrt{10}$ units
   (b) $\frac{1}{\sqrt{10}}$
   (c) $\frac{\sqrt{10}}{-3}$
   (d) -3
2. (a) $\frac{-\sqrt{21}}{5}$
   (b) -2
3. (a) $\frac{\sqrt{t^2+4}}{2}$
   (b) $\frac{2}{t}$
   (c) $\frac{4}{t^2+4}$
   (d) -1

End of chapter exercises

1. $1\frac{1}{2}$
2. 

   (a) $\frac{-3}{\sqrt{34}}$
   (b) -1
3. (a) $42{,}07°$
   (b) $63{,}43°$
   (c) $25°$
   (d) $45°$
4. (a) $a = 13{,}86$ cm
   (b) $b = 12{,}56$ cm
   (c) $c = 4{,}30$ cm
   (d) $d = 7{,}78$ cm
   (e) $e = 5{,}34$ cm
   (f) $f = 9{,}20$ cm
   (g) $g = 1{,}65$ cm
5. (a) $17{,}32$ cm
   (b) 10 cm
   (c) $25{,}08°$

6. $19{,}47°$
7. $53{,}54°$
8. $11{,}88$ units
9. $33{,}69°$
10. $23{,}96°$
11. 8,35 mm; 9,96 mm
12. **(a)** 5 cm
    (b) 4,83 cm; 1,29 cm
13. **(a)** $18°$
    (b) $23°$
14. $97{,}12°$
15. 5,65 cm; 8,70 cm

8. Analytical geometry

EMAFC

Exercise 8-1

1. **(a)** $\sqrt{29}$
   (b) $\sqrt{52}$
   (c) $\sqrt{17}$
2. **(a)** $x = 3$ or $x = 9$
   (b) $y = 3$ or $y = -5$

Exercise 8-2

1. **(a)** 1
   (b) $\frac{11}{8}$
   (c) $\frac{4}{3}$
2. **(a)** $\frac{-10}{3}$
   (b) 5

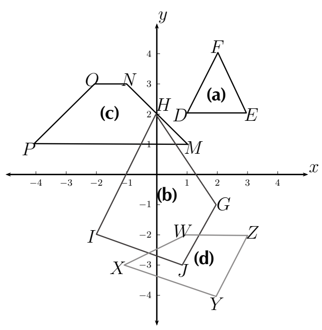
Exercise 8-3

1. **(a)** $AB \parallel CD$
   (b) Neither
   (c) Neither
2. **(a)** On same line
   (b) On same line
   (c) Not on same line
3. $\frac{1}{2}$
4. 6

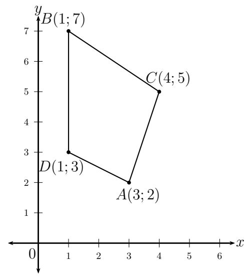
Exercise 8-4

1. **(a)** $(-1; 6)$
   (b) $(14; 32)$
   (c) $(\frac{2x-3}{2}; \frac{2y-5}{2})$
2. $P(8; 13)$
3. $S(4; -5)$

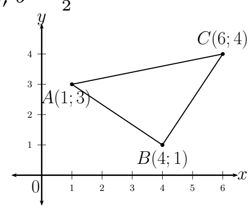
End of chapter exercises



1.
2. **(a)** $FG = \sqrt{26}; IH = \sqrt{41}; GH = \sqrt{8}; FI = \sqrt{29}$
   (b) No
   (c) No
   (d) Ordinary quadrilateral



3. **(a)**
   (b) $AB = \sqrt{10}; BC = \sqrt{13}; CD = 4; DA = \sqrt{5}$
5. **(b)** **i.** $\sqrt{10}$
   ii. 3
   (c) Trapezium
6. $H(3; 3)$
7. **(a)** $\sqrt{34}$
   (b) $\frac{1}{3}$
   (c) $(\frac{3}{2}; -\frac{1}{2})$
8. $a = 0$; $b = \frac{9}{2}$



9. **(a)**
   (c) $(\frac{7}{2}; \frac{7}{2})$
   (d) $\frac{-2}{3}$
10. **(a)** $y = \frac{1}{3}x + 3$
    (b) $\sqrt{40}$

9. Statistics

EMAFD

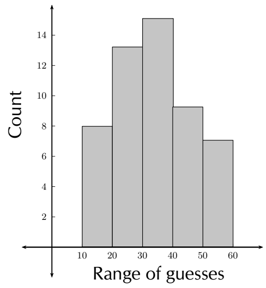
Exercise 9-1

1. (a) Mean = 13,2; Median = 11; Mode = 8
   (b) Mean = 26; Median = 25; Mode = 24
   (c) Mean = 11,2; Median = 11; Mode = 11
   (d) Mean = 34,29; Median = 32; No mode
2. Mean = 38,3; Median = 38; Mode = 33 and 42
3. (a) 5
   (b) 7

Exercise 9-2

1. (a)

Group	Freq
11 – 20	6
21 – 30	13
31 – 40	15
41 – 50	9
51 – 60	7

   (b) 

Exercise 9-3

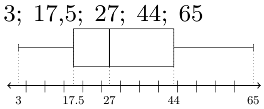
1. 53; $50 < m \leq 55$; $50 < m \leq 55$
2. 71,66; $65 < t \leq 75$; $65 < t \leq 75$
3. (a) $700 < x \leq 800$
   (b) 33 600
   (c) 700
   (d) 750
   (e) R 588 000

Exercise 9-4

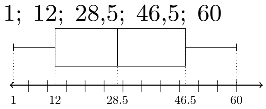
1. 9
2. $Q_1 = 6{,}2$; $Q_2 = 18$; $Q_3 = 29$
3. R = 70; $Q_1 = 41{,}5$; $Q_2 = 49{,}5$; $Q_3 = 66{,}5$; IQR = 25
4. (a) 15; 9; 16
   (b) 12; 7,5; 15,5
   (c) 10; 8; 7,5
   (d) 5; 4; 3,75
   (e) 14; 12; 19
   (f) 22; 15,5; 23

Exercise 9-5

1. 3; 17,5; 27; 44; 65

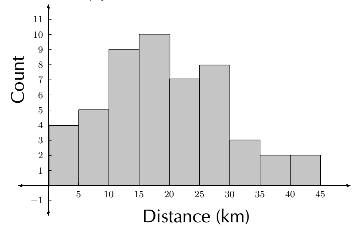


2. 1; 12; 28,5; 46,5; 60

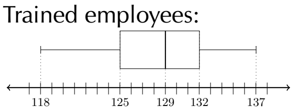


3. 3; 5; 7; 13; 16
4. (a) 15; 22; 25; 28; 35
   (b) 88; 92; 98; 100; 101

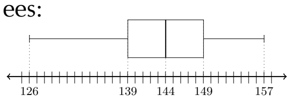
End of chapter exercises

1. 44
2. 6
3. (a) Mean and Mode
   (b) Bike 1= 1,02 s and Bike 2 = 1,0 s
   (c) Bike 2
4. (a) 19,9
   (b) i. 38%
       ii. 14%
       iii. 50%
   (c) 

5. (a) 129; 144
   (b) 7; 10
   (c) Trained employees:



   Untrained employees:



6. (a) R 182 222,22
   (b) R 100 000
   (c) R 100 000

10. Probability

EMAFE

Exercise 10-1

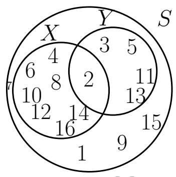
1. (a) $\frac{1}{2}$
   (b) $\frac{1}{3}$
   (c) $\frac{5}{6}$
   (d) $\frac{1}{3}$
2. (a) $\frac{1}{52}$
   (b) $\frac{1}{2}$
   (c) $\frac{3}{13}$

(d) $\frac{1}{13}$

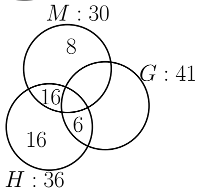
(e) $\frac{3}{13}$

3. $\frac{1}{5}$

Exercise 10-2

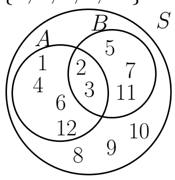


1.



2. (a) $H : 36$
   (b) 6
   (c) 29
   (d) 2

3. (a) $\{1; 2; 3; 4; 5; 6; 7; 8; 9; 10; 11; 12\}$
   (b) $\{1; 2; 3; 4; 6; 12\}$
   (c) $\{2; 3; 5; 7; 11\}$



   (d)
   (e) i. 12
       ii. 6
       iii. 5

Exercise 10-3

1. (a) 0,21
   (b) 0,64
   (c) 0
   (d) 0,72

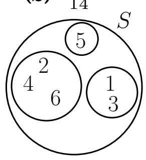
2. (a) 0,5
   (b) 0,23
   (c) 0,67
   (d) 0
   (e) 0,67
   (f) 0,67

3. (a) 0,11
   (b) 0,33
   (c) 0,16
   (d) 0,01
   (e) 0,80
   (f) 0,55
   (g) 0,16
   (h) 0,01

End of chapter exercises

1. 0,18

2. (a) $\frac{1}{6}$
   (b) $\frac{3}{14}$



3. (a) $\frac{1}{6}$
   (b) $\frac{1}{2}$
   (c) $\frac{1}{3}$

4. $\frac{5}{12}$

5. (a) $\frac{19}{30}$
   (b) $\frac{11}{30}$

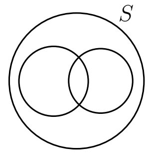
6. (a) $\frac{4}{9}$
   (b) $\frac{5}{9}$
   (c) $\frac{1}{9}$
   (d) $\frac{8}{9}$
   (e) $\frac{5}{9}$
   (f) $\frac{4}{9}$

7. (a) $\frac{11}{21}$
   (b) $\frac{11}{14}$

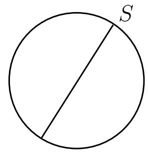
8. (a) $\frac{3}{56}$
   (b) $\frac{53}{56}$

9. (a) 103
   (b) i. $\frac{19}{30}$
       ii. $\frac{58}{103}$
   (c) $\frac{14}{19}$

10. (a) i. $\frac{2}{5}$
        ii. $\frac{3}{5}$
        iii. $\frac{1}{7}$
    (b) 10
    (c) 4

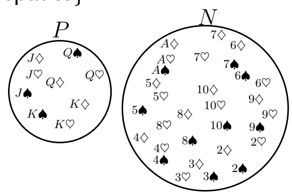


11. (a)

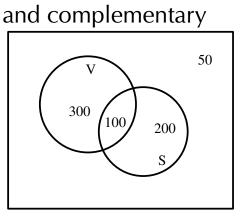


    (b)

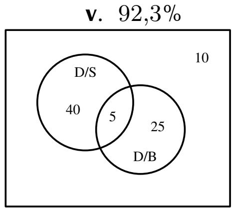
13. (a) {deck of cards without clubs}
    (b) $P = \{J;\ Q;\ K$ of hearts, diamonds or spades}
    (c) $N = \{A;\ 2;\ 3;\ 4;\ 5;\ 6;\ 7;\ 8;\ 9;\ 10$ of hearts, diamonds or spades}



    (d)
    (e) Mutually exclusive and complementary



14. (a)
    (b) i. 30,8%
        ii. 46,2%
        iii. 7,7
        iv. 15,4%
        v. 92,3%



15. (a) 50%
    (b) 31,25%
    (c) 6,25%

11. Euclidean Geometry

EMAFF

Exercise 11-1

1. $a = 138°$; $b = 42°$; $c = 138°$; $d = 138°$; $e = 42°$; $f = 138°$; $g = 42°$
2. $\hat{B}_1 = 110°$; $\hat{C}_1 = 80°$; $\hat{C}_2 = 30°$; $\hat{C}_3 = 70°$; $\hat{D}_1 = 100°$; $\hat{F}_1 = 70°$; $\hat{F}_2 = 30°$; $\hat{F}_3 = 80°$; $\hat{G}_1 = 70°$; $\hat{G}_2 = 30°$; $\hat{G}_3 = 80°$
3. $70°$

Exercise 11-2

1. **(a)** $x = y = 72°$
   (b) $x = 98°$
   (c) $x = 44°$; $y = 112°$
   (d) $x = 19$ units
   (e) $x = 25$ units
   (f) $x = 18$ units; $y = 4$ units
   (g) $x = 12$ units; $y = 13$ units

Exercise 11-6

1. **(a)** $x = 14$ units
   (b) $x = 3{,}5$ units
   (c) $x = 5$ units
   (d) $x = 28$ units; $y = 80°$
   (e) $x = 24°$;$y = 12$ units

End of chapter exercises

1. **(a)** Straight angle
   (b) Obtuse angle
   (c) Acute angle
   (d) Right angle
   (e) Reflex angle
   (f) Obtuse angle
   (g) Straight angle
   (h) Reflex angle
2. **(a)** F
   (b) T
   (c) T
   (d) F
   (e) T
   (f) T
   (g) T
   (h) F
   (i) F
3. **(a)** $x = 25°$
   (b) $x = 145°$
   (c) $x = 10$ units; $y = 12{,}5$ units
   (d) $x = 60°$
   (e) $x = 36°$
   (f) $x = 6$ units; $y = 10$ units
5. **(a)** $a = 107°$; $b = 73°$; $c = 107°$; $d = 73°$
   (b) $a = 80°$; $b = 80°$; $c = 80°$; $d = 80°$
   (c) $a = 50°$; $b = 45°$; $c = 95°$; $d = 85°$
7. **(a)** $x = 4{,}24$ cm
   (b) $x = 12$ cm
   (c) $x = 7{,}28$ cm
   (d) $x = 40$ mm
9. $x = 2{,}75$ units; $y = 30°$
10. $a = 5$ units; $b = 12$ units

12. Measurements

EMAFG

Exercise 12-1

1. 25 cm^2
2. 50 cm^2
3. 79 cm^2
4. 40 cm^2
5. 128 cm^2
6. $17{,}5$ cm^2
7. 60 cm
8. 525 cm^2

Exercise 12-2

1. **(a)** 344 cm^2
   (b) 471 cm^2
   (c) 533 cm^2
2. **(a)** $24\ \ell$
   (b) $22\ \ell$

Exercise 12-3

1. 420 cm^3
2. 500 cm^3
3. $785{,}4$ cm^3

Exercise 12-4

1. a. 282,7 cm^2
   b. 45,6 cm^2
   c. 180 cm^2
   d. 1 256,6 cm^2
2. a. 108,33 cm^3
   b. 270 cm^3
   c. 144 cm^3
   d. 4 188,8 cm^3
3. 175 cm^3; 190 cm^2

Exercise 12-5

1. Volume doubles
3. 31 552 cm^3; 96 112 cm^2

End of chapter exercises

1. (a) Cylinder= 352 cm^2
   Tri prism= 384 cm^2
   Rect prism= 72 cm^2
   (b) Cylinder= 502 cm^3
   Tri prism = 240 cm^3
   Rect prism= 40 cm^3
   (c) Cyl= 3166,7 cm^2
   Tri prism = 3 456 cm^2
   Rect prism= 684 cm^2
   (d) Cy= 13571,9 cm^3
   Tri prism = 6 480 cm^3
   Rect prism = 1 080 cm^3
2. (a) Cone= 225 cm^2
   Sq pyramid= 585 cm^2
   Half sphere= 100 cm^2
   (b) Cone= 94 cm^3
   Sq pyramid= 900 cm^3
   Half sphere= 134 cm^3
3. Volume= 335 103,2 cm^3
   S.A= 35 185,9 cm^3

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